Faculty of Engineering
Computer and Systems
Engineering Department

# CSE 372: Control Systems (2)

## Topic# 3

## Z-Transform

## Outline





## Introduction

- Z-transform in discrete time system analysis plays the same rule as the Laplace transform in continuous time system
- The Laplace transform converts a differential equation into algebraic equation in (s) for ease of analysis where the differential equation is used to describe the I/O relationship for continuous time system
- The Z-Transform converts a difference equation into algebraic equation in (z) for ease of analysis where the difference equation is used to describe the I/O relationship for discrete time system.

## Discrete time signal

- Discrete time signals arise if the system involves sampling operation of a continuous time signal.
  - The sampled signal is $x(0), x(T), x(2T), \cdots, x(kT)$ where $T$ is the sampling period.
- Discrete time signals also exists in discrete time system, or when a digital computer is used to process signals as a number sequence.
  - The number sequence is $x(0), x(1), x(2), \cdots, x(k)$.
- The Z-transform applies to
  - Continuous signal
  - Sampled signal
  - Number sequence



## Definition

- For continuous time signal and sampled signal:
  - $X(z) = Z\{x(t)\} = Z\{x(kT)\} = \sum_{k=0}^{\infty} x(kT) z^{-k}$.
- For a number sequence.
  - $X(z) = Z\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k}$.
- Where:
  - The variable z is a complex number
  - $x(t) = 0$ for $t < 0$
  - $x(k) = 0$ for $k < 0$



## Z-Transform of elementary functions

**Unit step function**

$$x(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \Rightarrow \quad \text{Sample time } T \quad \Rightarrow \quad x(k) = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\Downarrow$$

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad \Leftarrow \quad |z| > 1 \quad \Leftarrow \quad X(z) = 1 + z^{-1} + z^{-2} + \cdots$$

**Unit ramp function**

$$x(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \Rightarrow \quad \text{Sample time } T \quad \Rightarrow \quad x(k) = \begin{cases} kT & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\Downarrow$$

$$X(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2} = \frac{Tz}{(z - 1)^2} \quad \Leftarrow \quad |z| > 1 \quad \Leftarrow \quad X(z) = T(z^{-1} + 2z^{-2} + \cdots)$$

## Z-Transform

Polynomial function

$$x(k) = a^k \quad \Rightarrow \quad X(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad |z| > a$$

Exponential function

$$x(k) = e^{-akT} \quad \Rightarrow \quad X(z) = 1 + e^{-aT} z^{-1} + \cdots = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}} \qquad |z| > e^{-aT}$$

Sinusoidal function

$$x(k) = \sin k\omega T = \frac{e^{jk\omega T} - e^{-jk\omega T}}{2j} \quad \Rightarrow \cdots \Rightarrow \quad X(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \qquad |z| > 1$$



## Properties of Z-Transform

Linearity. Let $X_1(z) = Z(x_1(k))$, $X_2(z) = Z(x_2(k))$, $\alpha_1 \in I\!R$ and $\alpha_2 \in I\!R$. Then

$$Z(\alpha_1 x_1(k) + \alpha_2 x_2(k)) = \alpha_1 X_1(z) + \alpha_2 X_2(z).$$

Multiplication by $a^k$. Let $X(z) = Z(x(k))$ and $a \in \mathbb{C}$. Then

$$Z(a^k x(k)) = X\left(\frac{z}{a}\right).$$

Shifting Theorem. Let $X(z) = Z(x(k))$, $n \in I\!N$ and $x(k) = 0$, for $k < 0$. Then

$$Z(x(k-n)) = z^{-n} X(z).$$

In addition

$$Z(x(k+n) = z^n \left[ X(z) - \sum_{k=0}^{n-1} x(k) z^{-k} \right].$$

Note that $x(k+n)$ is the sequence shifted to the left (with a forward time shift), and $x(k-n)$ is the sequence shifted to the right (with a backward time shift).



## Properties of Z-Transform

Backward difference. The (first) backward difference between $x(k)$ and $x(k-1)$ is defined as

$$\nabla x(k) = x(k) - x(k-1).$$

Then

$$Z(\nabla x(k)) = Z(x(k)) - Z(x(k-1)) = X(z) - z^{-1} X(z) = (1 - z^{-1}) X(z).$$

Forward difference. The (first) forward difference between $x(k+1)$ and $x(k)$ is defined as

$$\Delta x(k) = x(k+1) - x(k).$$

Then

$$Z(\Delta x(k)) = Z(x(k+1)) - Z(x(k)) = (zX(z) - zx(0)) - X(z) = (z-1)X(z) - zx(0).$$

## Properties of Z-Transform

Complex translation Theorem. Let $X(z) = Z(x(k))$ and $\alpha \in \mathbb{C}$. Then

$$Z(e^{-\alpha k}x(k)) = X(ze^{\alpha}).$$

Initial value Theorem. Let $X(z) = Z(x(k))$ and suppose that

$$\lim_{z\to\infty} X(z)$$

exists. Then

$$x(0) = \lim_{z\to\infty} X(z).$$

Final value Theorem. Let $X(z) = Z(x(k))$ and suppose that all poles of $X(z)$ are in $D^-$ ($D^-$ denotes the interior of the unity circle), with the possible exception of a single pole at $z = 1$. Then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z).$$

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## Properties of Z-Transform

Complex differentiation. Let $X(z) = Z(x(k))$. Then

$$Z(kx(k)) = -z\frac{d}{dz}X(z),$$

and the derivative $\frac{d}{dz}X(z)$ converges in the same region as $X(z)$.

Complex integration. Let $X(z) = Z(x(k))$ and $g(k) = \frac{x(k)}{k}$. Assume $\lim_{k\to 0} g(k)$ is finite. Then

$$Z(g(k)) = \int_z^{\infty} \frac{X(\zeta)}{\zeta} d\zeta + \lim_{k\to 0} g(k)$$

Real convolution Theorem. Let $X_1(z) = Z(x_1(k))$ and $X_2(z) = Z(x_2(k))$. Then

$$X_1(z)X_2(z) = Z\left(\sum_{h=0}^{k} x_1(h)x_2(k-h)\right) = Z\left(\sum_{h=0}^{k} x_1(k-h)x_2(h)\right).$$

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## Properties of Z-Transform

| | Time domain | Z-domain |
|---|---|---|
| Notation | $x[n] = \mathcal{Z}^{-1}\{X(z)\}$ | $X(z) = \mathcal{Z}\{x[n]\}$ |
| Linearity | $a_1x_1[n] + a_2x_2[n]$ | $a_1X_1(z) + a_2X_2(z)$ |
| Time shifting | $x[n-k]$ | $z^{-k}X(z)$ |
| Scaling in the z-domain | $a^n x[n]$ | $X(a^{-1}z)$ |
| Time reversal | $x[-n]$ | $X(z^{-1})$ |
| Conjugation | $x^*[n]$ | $X^*(z^*)$ |
| Real part | $\mathrm{Re}\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ |
| Imaginary part | $\mathrm{Im}\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ |
| Differentiation | $nx[n]$ | $-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$ |
| Convolution | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ |
| Multiplication | $x_1[n]x_2[n]$ | $\frac{1}{j2\pi}\oint_C X_1(v)X_2(\frac{z}{v})v^{-1}\mathrm{d}v$ |
| Parseval's relation | $\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$ | $\frac{1}{j2\pi}\oint_C X_1(v)X_2^*(\frac{1}{v^*})v^{-1}\mathrm{d}v$ |

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## Inverse of Z-Transform

- When using the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} (x[n] z^{-n})$$

it is often useful to be able to find x [n] given X (z) (inverse transform)

- There are *at least* 4 different methods to do this:
  1. Inspection
  2. Partial-Fraction Expansion
  3. Power Series Expansion
  4. Long Division



## Inverse of Z-Transform

### Inspection Method

- This "method" is to basically become familiar with the z-transform pair tables and then "reverse engineer"

**Example**

When given

$$X(z) = \frac{z}{z-\alpha}$$

we could determine "by inspection" that

$$x[n] = \alpha^n u[n]$$



## Inverse of Z-Transform

### Common Z Transform Pairs

| | Signal, $x[n]$ | Z-transform, $X(z)$ | ROC |
|---|---|---|---|
| 1 | $\delta[n]$ | 1 | all $z$ |
| 2 | $\delta[n-n_0]$ | $\frac{1}{z^{n_0}}$ | $\|z\| > 0$ |
| 3 | $u[n]$ | $\frac{z}{z-1}$ | $\|z\| > 1$ |
| 4 | $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| > \|a\|$ |
| 5 | $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |

## Inverse of Z-Transform

| | | | |
|---|---|---|---|
| 6 | $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| < \|a\|$ |
| 7 | $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| 8 | $\cos(\omega_0 n)u[n]$ | $\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$ | $\|z\| > 1$ |
| 9 | $\sin(\omega_0 n)u[n]$ | $\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$ | $\|z\| > 1$ |
| 10 | $a^n \cos(\omega_0 n)u[n]$ | $\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$ | $\|z\| > \|a\|$ |
| 11 | $a^n \sin(\omega_0 n)u[n]$ | $\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$ | $\|z\| > \|a\|$ |



## Inverse of Z-Transform

- Partial fraction method:
  - In general, one can find the *partial fraction expansion* for the *z*-transform expressions that are rational functions of *z*
    We will have:
    $$X(z)=\sum_{i=1}^{m}\frac{A_i}{1-a_i z^{-1}}$$
  - One can then take the inverse *z*-transform of *each individual term* very easily
- The values of $A_i$ can be calculated from: $A_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}$



## Inverse of Z-Transform

- If X(z) or X(Z)/z has multiple poles at z=$p_1$ then :

$$\frac{X(z)}{z}=\frac{c_1}{(z-p_1)^2}+\frac{c_2}{z-p_1}$$

The coefficients $c_1$ and $c_2$ are determined from

$$c_1=\left[(z-p_1)^2\frac{X(z)}{z}\right]_{z=p_1}$$

$$c_2=\left\{\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]\right\}_{z=p_1}$$

## Inverse of Z-Transform

- Find the inverse z-transform of

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + (-3z^{-1}) + 2z^{-2}}$$

In this case $M = N = 2$, so we have to use long division to get

$$X(z) = \frac{1}{2} + \frac{\frac{1}{2} + \frac{7}{2}z^{-1}}{1 + (-3z^{-1}) + 2z^{-2}}$$

- Next factor the denominator

$$X(z) = \frac{1}{2} + \frac{\frac{1}{2} + \frac{7}{2}z^{-1}}{(1 - 2z^{-1})(1 - z^{-1})}$$

- Now do partial-fraction expansion

$$X(z) = \frac{1}{2} + \frac{A_1}{1 - 2z^{-1}} + \frac{A_2}{1 - z^{-1}} = \frac{1}{2} + \frac{\frac{9}{2}}{1 - 2z^{-1}} + \frac{-4}{1 - z^{-1}}$$

- Now each term can be inverted using the inspection method and the z-transform table.

$$x[n] = \frac{1}{2}\delta[n] + \frac{9}{2}2^n u[n] + (-4u[n])$$



## Inverse of Z-Transform

*Example.*

$$X(z) = \frac{10z + 5}{(z-1)(z-1/5)}$$

$$\Downarrow$$

$$\frac{X(z)}{z} = \frac{10z + 5}{z(z-1)(z-1/5)} = 25\frac{1}{z} + \frac{75}{4}\frac{1}{z-1} - \frac{175}{4}\frac{1}{z-1/5}$$

$$\Downarrow$$

$$X(z) = 25 + \frac{75}{4}\frac{z}{z-1} - \frac{175}{4}\frac{z}{z-1/5} = 25 + \frac{75}{4}\frac{1}{1-z^{-1}} - \frac{175}{4}\frac{1}{1-1/5z^{-1}}$$

$$\Downarrow$$

$$x(k) = 25\delta(k) + \frac{75}{4}1^k - \frac{175}{4}(1/5)^k \qquad k \geq 0$$



## Inverse of Z-Transform

*Example.*

$$X(z) = X(z) = \frac{3z + 1}{z^2 - z + 1/2}$$

$$\Downarrow$$

$$\frac{X(z)}{z} = \frac{3z+1}{z(z-(1/2+1/2j))(z-(1/2-1/2j))} = \frac{10}{z} - \frac{5+8j}{z-(1/2+1/2j)} - \frac{5-8j}{z-(1/2-1/2j)}$$

$$\Downarrow$$

$$X(z) = 10 - \frac{(5+8j)z}{z-(1/2+1/2j)} - \frac{(5-8j)z}{z-(1/2-1/2j)} = 10 - \frac{5+8j}{1-(1/2+1/2j)z^{-1}} - \frac{5-8j}{1-(1/2-1/2j)z^{-1}}$$

$$\Downarrow$$

$$x(k) = 10\delta(k) - 10\left(\frac{1}{\sqrt{2}}\right)^k \cos\frac{k\pi}{4} + 16\left(\frac{1}{\sqrt{2}}\right)^k \sin\frac{k\pi}{4} \qquad k \geq 0$$

## Inverse of Z-Transform

Power Series Expansion

- One can find the inverse **z**-transform of non-rational expressions of **z**, by writing that expression as a power series (for example using Taylor expansion)
- The **z**-transform is defined as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x[n] z^{-n}\right)$$

- Then each term of the sequence x [n] can be determined by looking at the coefficients of the respective power of **z**$^{-n}$

**Example** Now look at the z-transform of a **finite-length sequence.**

$$\begin{aligned} X(z) &= z^2\left(1+2z^{-1}\right)\left(1-\tfrac{1}{2}z^{-1}\right)\left(1+z^{-1}\right) \\ &= z^2+\tfrac{5}{2}z+\tfrac{1}{2}+\left(-\left(z^{-1}\right)\right) \end{aligned}$$

In this case, since there were no poles, we multiplied the factors of $X(z)$. Now, by inspection, it is clear that

$$x[n] = \delta[n+2] + \frac{5}{2}\delta[n+1] + \frac{1}{2}\delta[n] + \left(-\left(\delta[n-1]\right)\right)$$



## Inverse of Z-Transform

- Using the power series expansion, find the inverse z-transform of the following expression:

$$X(z) = e^{(a/z)}, \quad |z| > |a|$$

- **Solution:** With $|z| > |a|$ or equivalently $\left|az^{-1}\right| < 1$, we can use the Taylor series expansion for $e^{(a/z)}$ as follows:

$$e^{(az^{-1})} = 1 + az^{-1} + \frac{a^2 z^{-2}}{2!} + \cdots, \quad \left|az^{-1}\right| < 1$$

This implies that:

$$x[n] = \begin{cases} \dfrac{a^n}{n!}, & n \ge 0 \\ 0, & n < 0 \end{cases}$$



## Inverse of Z-Transform

**Example** Suppose

$$X(z) = \log_n\left(1 + \alpha z^{-1}\right)$$

Noting that

$$\log_n(1+x) = \sum_{n=1}^{\infty}\left(\frac{-1^{n+1} x^n}{n}\right)$$

Then

$$X(z) = \sum_{n=1}^{\infty}\left(\frac{-1^{n+1} \alpha^n z^{-n}}{n}\right)$$

Therefore

$$x[n] = \begin{cases} \frac{-1^{n+1}\alpha^n}{n} \text{ if } n \ge 1 \\ 0 \text{ if } n \le 0 \end{cases}$$

# Inverse of Z-Transform

## Long Division Method

- In algebra, ***polynomial long division*** is an algorithm for dividing a polynomial by another polynomial of lower degree
- A generalized version of the familiar arithmetic technique called ***long division***
- It can be done by hand, because it separates an otherwise complex division problem into smaller ones
- For any polynomials *F(z)* and *G(z)*, where the degree of *F(z)* is greater than or equal to the degree of *G(z)*, there exist unique polynomials *Q(z)* and *R(z)* such that

$$\frac{F(z)}{G(z)} = Q(z) + \frac{R(z)}{G(z)} \quad \Leftrightarrow \quad G(z)\overline{\big)F(z)}^{\,Q(z)} \;\; \text{remainder } R(z)$$

with *R(z)* having smaller degree than *G(z)*



# Inverse of Z-Transform

## Long Division Method

Using long division, find the inverse z-transform of $X(z) = \frac{1}{1 - az^{-1}}$

with the regions of convergence: (a) $|z| > |a|$ (b) $|z| < |a|$

**Solution:**

(a) We have:

$$\begin{array}{r l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ 1 - az^{-1} \big) & 1 \\ & \underline{1 - az^{-1}} \\ & az^{-1} \\ & \underline{az^{-1} - a^2z^{-2}} \\ & a^2z^{-2} \end{array}$$

This means that: $\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$

Since $|z| > |a|$, we have $|az^{-1}| < 1$ and the series $1 + az^{-1} + a^2z^{-2} + \cdots$ converges. So, by comparing this series with the general equation for the z-transform of a signal we will have: $x[n] = 0$ for $n < 0$, $x[0] = 1$, $x[1] = a$, $x[2] = a^2, \cdots$

and in general: $x[n] = a^n u[n]$



# Inverse of Z-Transform

(b) For $|z| < |a|$ the series given by $\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$ does not converge as $|az^{-1}| > 1$

Therefore, one can use the following form of long division:

$$\begin{array}{r l} & -a^{-1}z - a^{-2}z^2 - \cdots \\ -az^{-1} + 1 \big) & 1 \\ & \underline{1 - a^{-1}z} \\ & a^{-1}z \end{array}$$

This means that: $\frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 - \cdots$

Since $|z| < |a|$, we have $|a^{-1}z| < 1$ and the series converges

So, by comparing this series with the general equation for the z-transform of a signal we will have: $x[n] = 0$ for $n \geq 0$, $x[-1] = -a^{-1}$, $x[-2] = -a^{-2}, \cdots$ and in general: $x[n] = -a^n u[-n-1]$

## Summary

| | f(t) | F(s) | F(z) | f(kT) |
|---|---|---|---|---|
| 1. | $u(t)$ | $\frac{1}{s}$ | $\frac{z}{z-1}$ | $u(kT)$ |
| 2. | $t$ | $\frac{1}{s^2}$ | $\frac{Tz}{(z-1)^2}$ | $kT$ |
| 3. | $t^n$ | $\frac{n!}{s^{n+1}}$ | $\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$ | $(kT)^n$ |
| 4. | $e^{-at}$ | $\frac{1}{s+a}$ | $\frac{z}{z-e^{-aT}}$ | $e^{-akT}$ |
| 5. | $t^n e^{-at}$ | $\frac{n!}{(s+a)^{n+1}}$ | $(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$ | $(kT)^n e^{-akT}$ |
| 6. | $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ | $\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$ | $\sin \omega kT$ |
| 7. | $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ | $\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$ | $\cos \omega kT$ |
| 8. | $e^{-at}\sin \omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ | $\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$ | $e^{-akT}\sin \omega kT$ |
| 9. | $e^{-at}\cos \omega t$ | $\frac{s+a}{(s+a)^2+\omega^2}$ | $\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$ | $e^{-akT}\cos \omega kT$ |

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## Summary

| | Theorem | Name |
|---|---|---|
| 1. | $z\{af(t)\} = aF(z)$ | Linearity theorem |
| 2. | $z\{f_1(t)+f_2(t)\} = F_1(z)+F_2(z)$ | Linearity theorem |
| 3. | $z\{e^{-at}f(t)\} = F(e^{aT}z)$ | Complex differentiation |
| 4. | $z\{f(t-nT)\} = z^{-n}F(z)$ | Real translation |
| 5. | $z\{tf(t)\} = -Tz\frac{dF(z)}{dz}$ | Complex differentiation |
| 6. | $f(0) = \lim_{z\to\infty} F(z)$ | Initial value theorem |
| 7. | $f(\infty) = \lim_{z\to 1}(1-z^{-1})F(z)$ | Final value theorem |

Note: $kT$ may be substituted for $t$ in the table.

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