MCT321: Introduction to Nano-Mechatronics
Lectures #6&7: Simple Beam Theory

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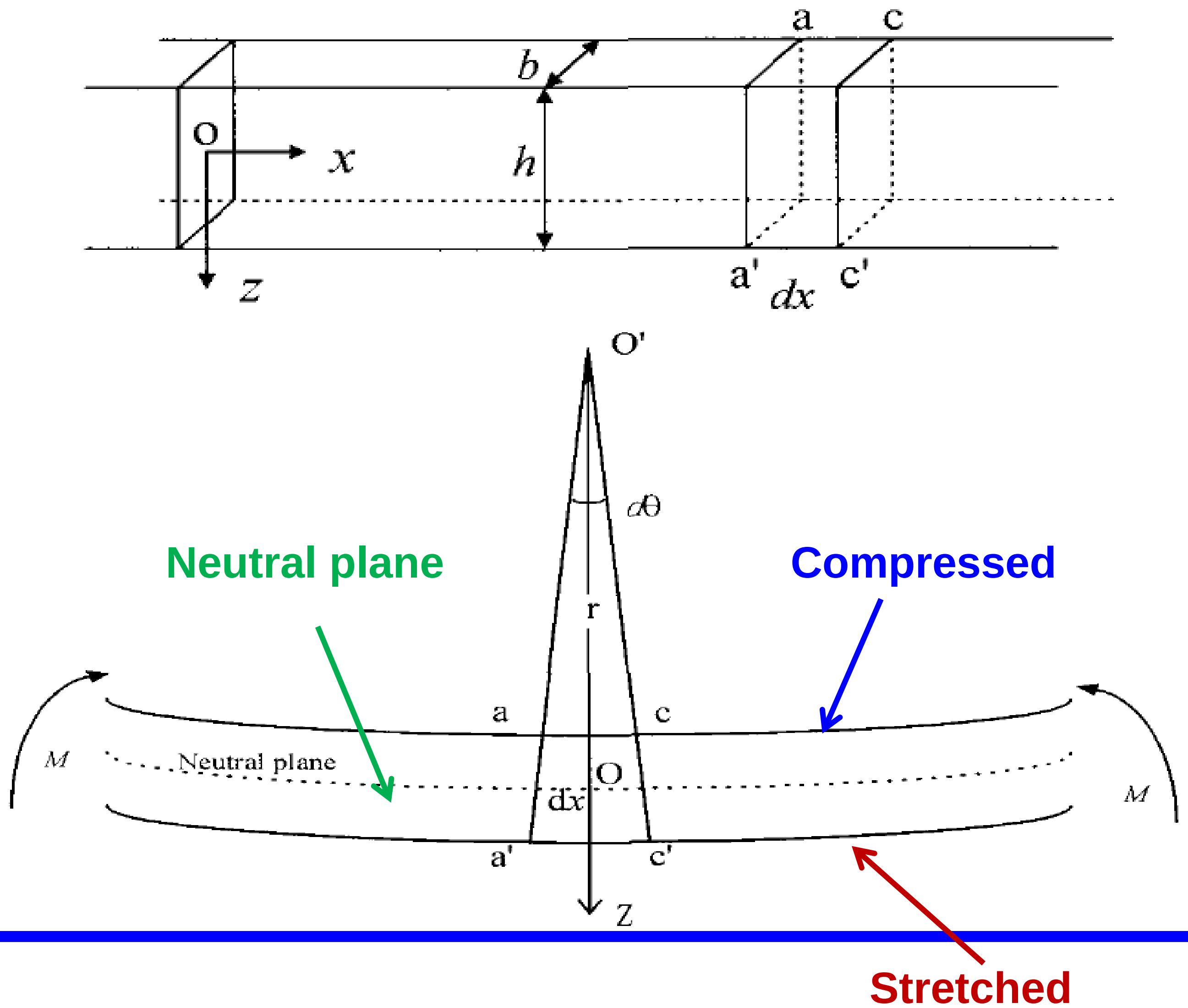
Outline

- Simple beam theory
- Microstructures
- Beam combinations

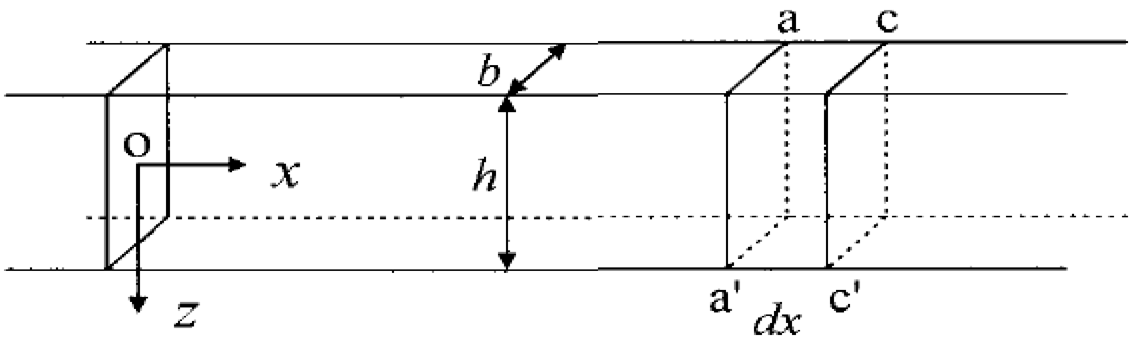
Simple beam theory



Simple beam theory

- Consider an elemental section *dx* between two vertical planes *aa'* and *cc'.*
- We limit our analysis here to **small deflections only and uniform cross section area**.
- Generally the displacement of the beam in the z-direction, *w*, is function of *x*, i.e. *w=w(x)*.
- ***w(x)* is the displacement function of the beam**.

➔ Our goal is to find *w(x)* for the beam



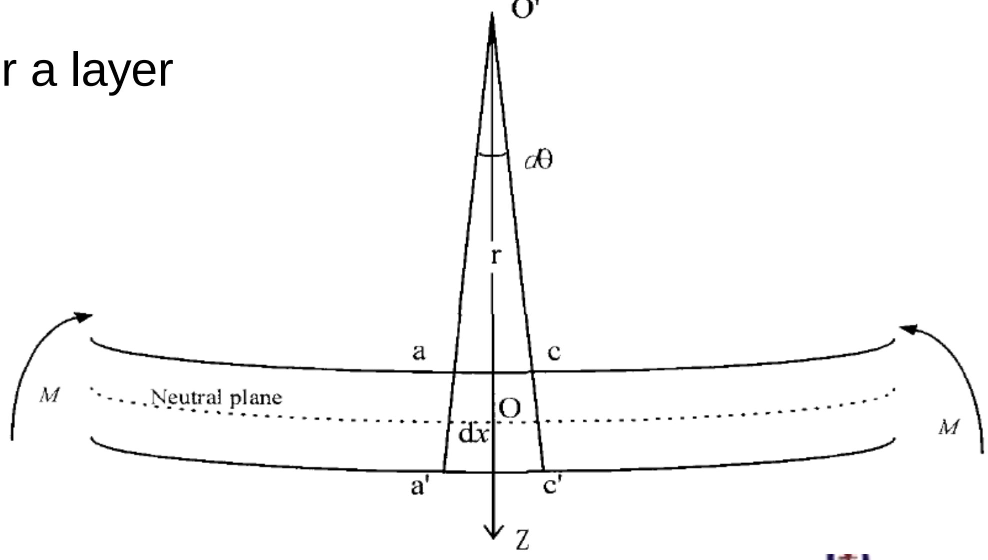
Simple beam theory

- If the radius of curvature of the elemental section *dx* at the central plane (z = 0) is OO' = *r*, then,

$$dx = rd\theta$$

- But, for a layer of beam away from the central plane ($z \neq 0$) the material is either stretched or compressed as a result of the bending moment *M.*
- The elongation in the x-direction for a layer at z (z > 0) is:

$$\Delta(dx) = (r+z)d\theta - rd\theta = zd\theta$$



Simple beam theory

- The strain is the relative elongation of the material

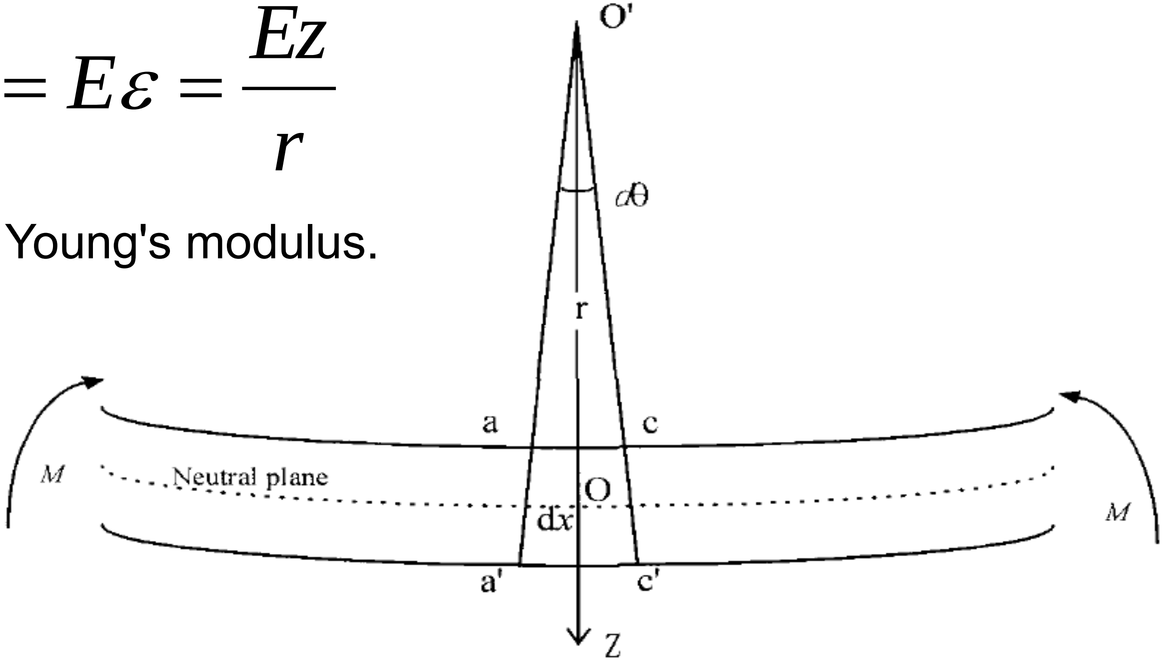
$$\varepsilon = \frac{\Delta(dx)}{dx} = \frac{z d\theta}{r d\theta} = \frac{z}{r}$$

- According to Hook's law,

$$T = E\varepsilon = \frac{Ez}{r}$$

- Where E is the material's Young's modulus.

Stretching, *T* is +ve
Compression, *T* is -ve



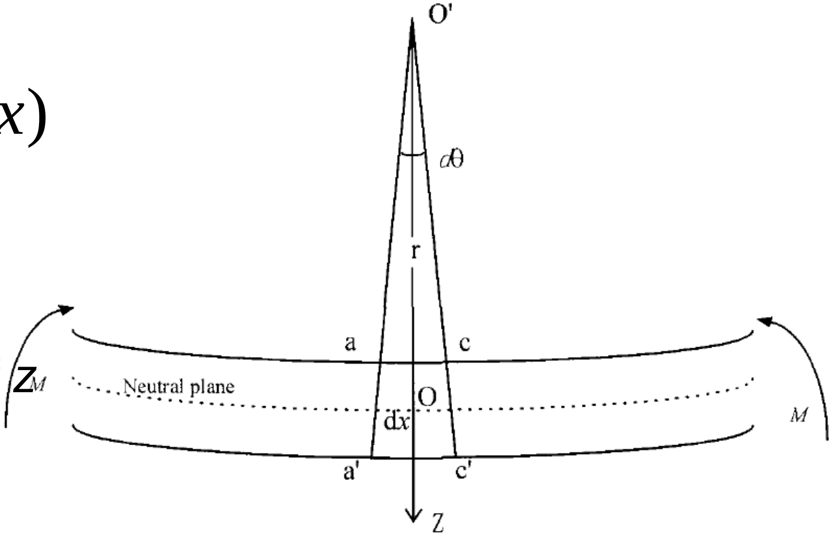
Simple beam theory

- From basic calculus, (only is true for small deflection approximation)

$$\frac{1}{r} = w''(x)$$

- Then

$$T = \frac{Ez}{r} = Ezw''(x)$$



- Always *w"(x)* is negative in this case,

  So, the sign of *T* is determined by the sign of *z*

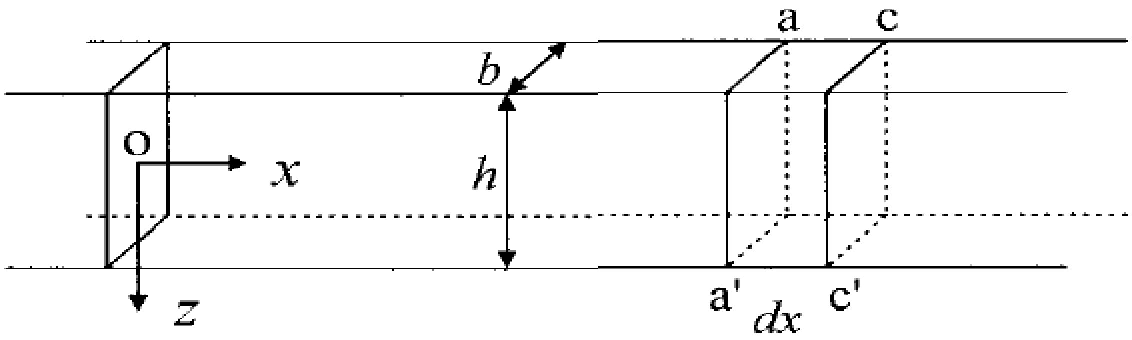
$$T(x) = \frac{Ez}{r} = -Ezw''(x)$$

Simple beam theory

- **Neutral plane for pure bending:**
- Axial force of a beam can be expressed as:

$$F = \int T(z) b dz = -Ebw''(x) \int_{-\frac{h}{2}}^{\frac{h}{2}} z dz$$

- This axial force must equal to zero at the neutral plane.

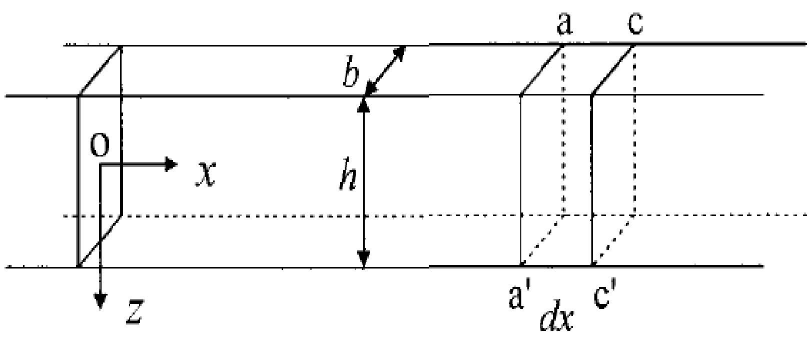


Simple beam theory

- **<u>Bending moment and moment of inertia:</u>**
- **The bending moment is**

$$M(x) = \int (z)T(z)dA = -Ew''(x)\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b dz$$

$$dA = bdz$$



Simple beam theory

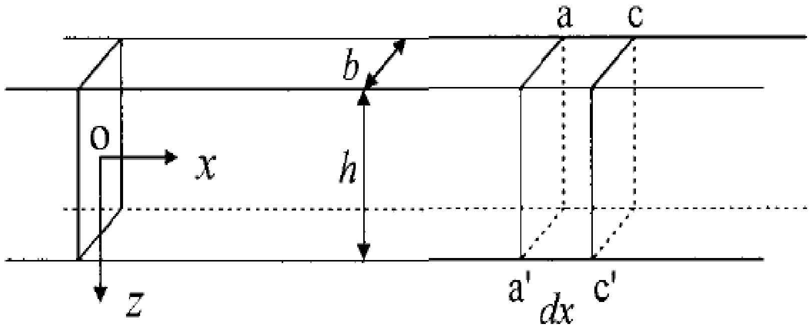
- **Bending moment and moment of inertia:**
- But, I is the area moment of inertia perpendicular to the z-axis,

$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 b(z) dz$$

- Then,

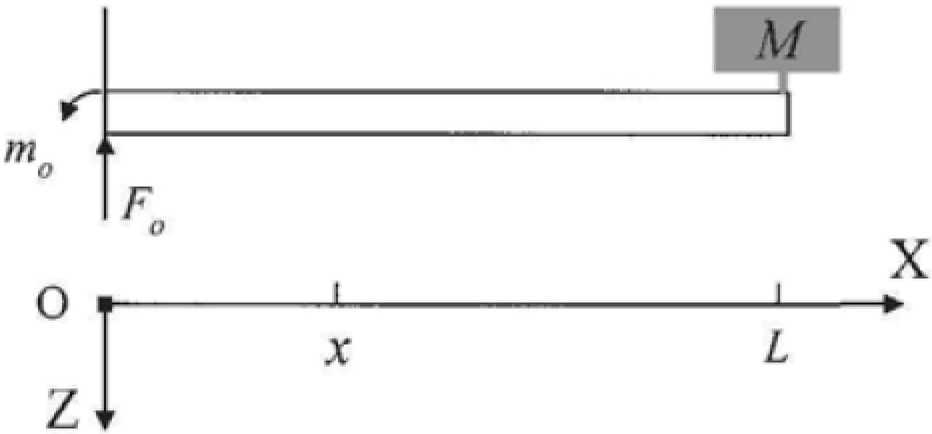
$$M(x) = -EIw''(x)$$

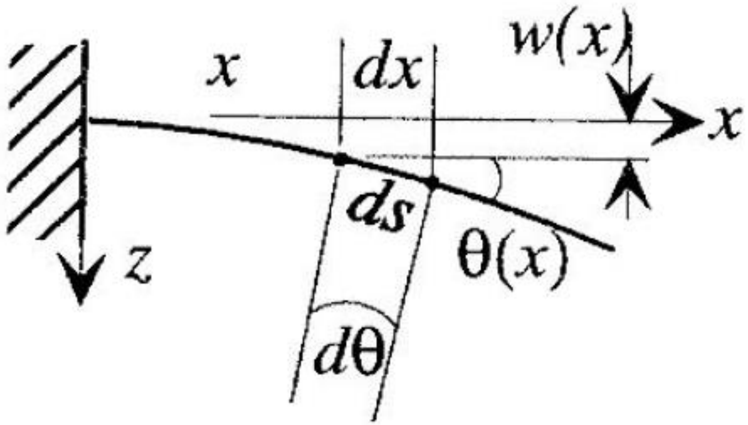
$$\frac{zM(x)}{I} = -Ezw''(x) = T$$



Microstructures

- **<u>Cantilever beam with concentrated end loading: (Free end beam)</u>**



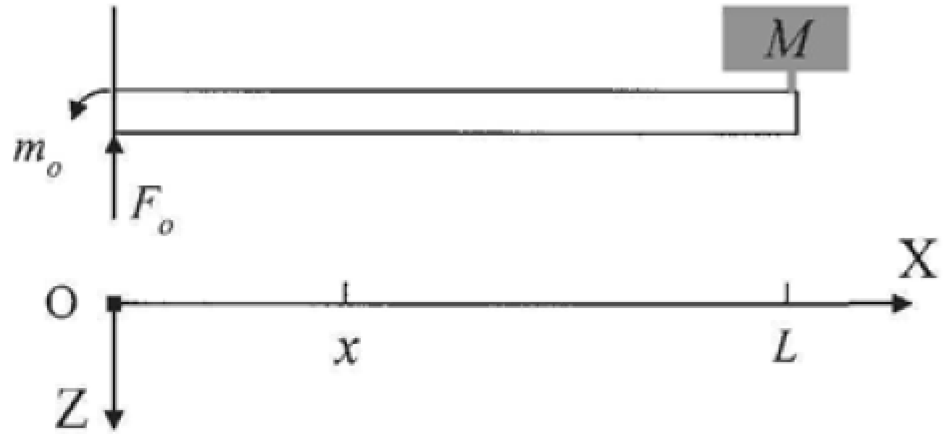


- The force, *F*, acting on the end of the beam is *Mg*.
- At equilibrium, $F_o = F = Mg$.
- Stress is **<u>maximum</u>** at $x = 0$ and **<u>ZERO</u>** at $x = L$ ➔➔➔ $w''(L) = 0$
- Boundary conditions:

$$w(0) = 0,\ w'(0) = 0,\ w''(L) = 0$$

Microstructures

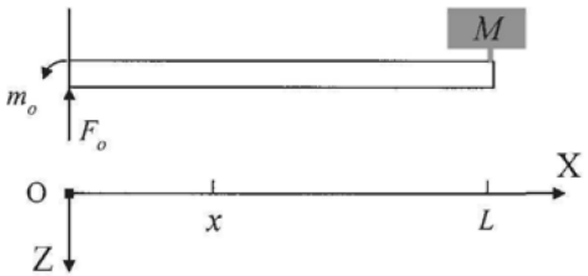
- **Cantilever beam with concentrated end loading: (Free end beam)**

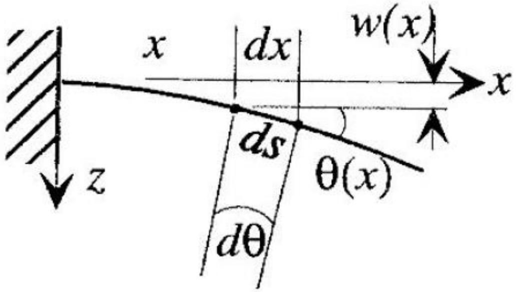


- At point x and from left hand side, $-EIw''(x) = M(x) = F_o x - m_o$
- Integrating: $-EIw'(x) = F_o \frac{x^2}{2} - m_o x + C_1$
- Applying BCs.: $w(0) = 0, w'(0) = 0, w''(L) = 0$ ➔ $C_1 = 0$
- Integrating: $-EIw(x) = F_o \frac{x^3}{6} - m_o \frac{x^2}{2} + C_2$
- Applying BCs.: ➔ $C_2 = 0$

Microstructures

- **Cantilever beam with concentrated end loading: (Free end beam)**





- To find m_o, $w''(L)=0 ==> 0=F_oL-m_o ==> m_o=F_oL=FL$
- So the displacement function is ($F_o = F$), $-EIw(x)=F\dfrac{x^3}{6}-FL\dfrac{x^2}{2}$
- Maximum bending (displacement) @ the free end ($x = L$)

$$\therefore w(x)=\frac{F}{EI}\left\{L\frac{x^2}{2}-\frac{x^3}{6}\right\}=\frac{Fx^2}{6EI}(3L-x)=\frac{Mgx^2}{6EI}(3L-x)$$

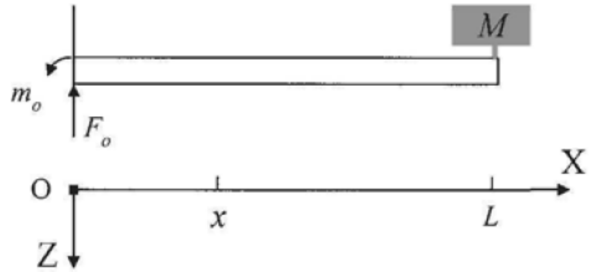
- Substitute with the value of the area moment of inertia,

$$I=\frac{bh^3}{12}$$

$$\therefore w(L)=\frac{MgL^2}{6EI}(3L-L)=\frac{MgL^2}{6EI}2L=\frac{MgL^3}{3EI}=\frac{4MgL^3}{Ebh^3}$$

Microstructures

- **<u>Cantilever beam with concentrated end loading: (Free end beam)</u>**



- To find beam stiffness (as a spring),

$$K = \frac{F}{w(L)} = \frac{Mg}{w(L)} = \frac{Ebh^3}{4L^3}$$

- To find the stress distribution we use,

$$T = -Ezw''(x) \quad \text{and} \quad -EIw''(x) = M(x)$$

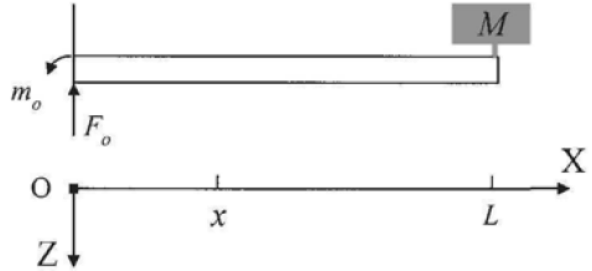
$$\therefore T = Ez\frac{M(x)}{EI} = z\frac{M(x)}{I} = z\frac{Fx - FL}{I}$$

- Maximum stress occurs at *x = 0* and *z=-h/2*

$$\therefore T_{\max}\Big|_{x=0} = \left(-\frac{h}{2}\right)\frac{-FL}{I} = \frac{FhL}{2I} = \frac{6FL}{bh^2} = \frac{6(Mg)L}{bh^2}$$

Microstructures

- **<u>Cantilever beam with concentrated end loading: (Free end beam)</u>**



- To find beam natural, or resonant, frequency:

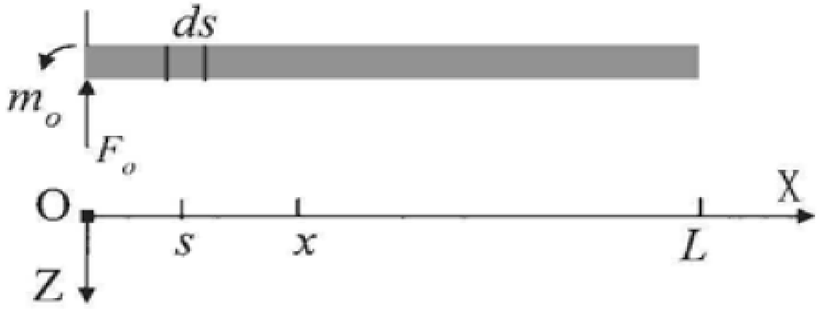
$$f_r = \frac{1}{2\pi}\sqrt{\frac{K}{M + M_{beam}}}$$

- Assume that $M >> M_{beam}$,

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{Ebh^3}{4L^3M}}$$

Microstructures

- **Cantilever beam bending under its weight: (Free end beam)**



- At point x and from left hand side,

$$-EIw''(x) = M(x) = F_o x - m_o - \int_0^x \frac{M_b g}{L}(x-s)ds$$

- Where, $M_b = \rho bhL$

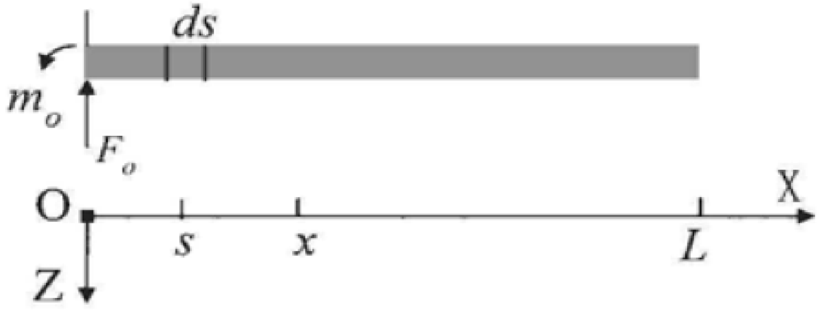
$$\therefore -EIw''(x) = F_o x - m_o - \frac{M_b g x^2}{2L}$$

$$\because F_o = M_b g$$

$$\therefore -EIw''(x) = M_b g x - m_o - \frac{M_b g x^2}{2L} \qquad (1)$$

Microstructures

- **Cantilever beam bending under its weight: (Free end beam)**



- Boundary Conditions are,

$$w(0) = 0, w'(0) = 0, w''(L) = 0 \qquad (2)$$

- Solving (1) and (2),

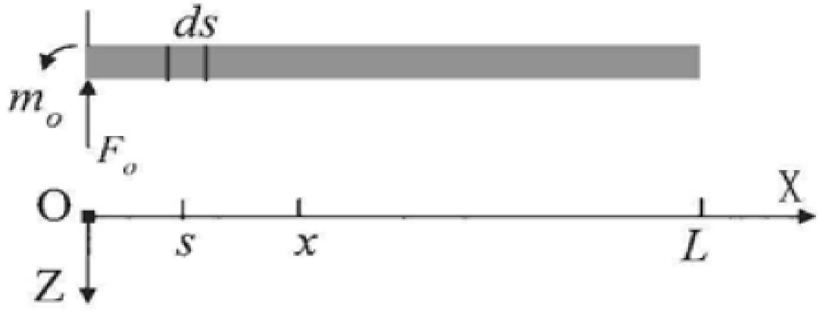
$$\therefore m_o = \frac{M_b g L}{2}$$

- The displacement function will be:

$$w(x) = \frac{M_b g x^2 (x^2 - 4Lx - 6L^2)}{24ELI} = \frac{M_b g x^2 (x^2 - 4Lx - 6L^2)}{2EbLh^3} \quad (3)$$

Microstructures

- **Cantilever beam bending under its weight: (Free end beam)**



- **Stress at the top surface of the beam:**
- First find w''(x) from equation (1).
- Second substitute in $T = -Ezw''(x)$
- z = -h/2

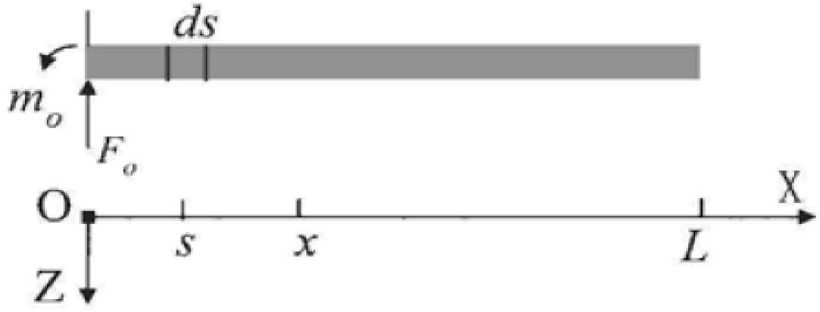
$$\therefore T(x) = -E\left(-\frac{h}{2}\right)w''(x) = \frac{M_b g (L-x)^2}{4LI} = \frac{3M_b g (L-x)^2}{bLh^2}$$

- Maximum stress at x = 0

$$\therefore T(0) = T_{max} = \frac{M_b g h L}{4I} = \frac{3M_b g L}{bh^2} = \frac{3\rho g L^2}{h}$$

Microstructures

- **Cantilever beam bending under its weight: (Free end beam)**



- **Resonant (natural) frequency:**
- First find *w(L)* from equation (3).

$$w(L) = w_{\max} = \frac{M_b g L^3}{8EI} = \frac{3M_b g L^3}{8Ebh^3} = \frac{3\rho g L^4}{2Eh^2}$$

- Second Find *K*

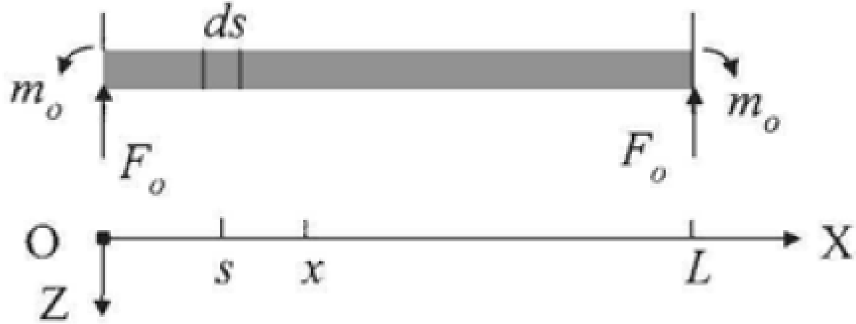
$$K = \frac{F}{w_{\max}} = \frac{M_b g}{w_{\max}} = \frac{2Ebh^3}{3L^3}$$

- Last find f_r

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K}{M_b}} = \frac{1}{2\pi}\sqrt{\frac{2Ebh^3}{3L^3 M_b}}$$

Microstructures

- **Double-clamped beam (bridge):**



- At point x and from left hand side,

$$-EIw''(x) = M(x) = F_o x - m_o - \int_0^x \frac{M_b g}{L}(x-s)\,ds$$

- Where, $M_b = \rho bhL$

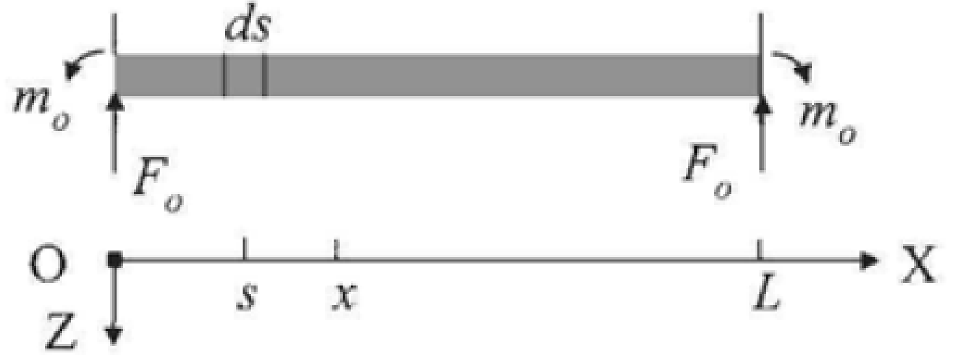
$$\therefore -EIw''(x) = F_o x - m_o - \frac{M_b g x^2}{2L}$$

$$\because F_o = \frac{M_b g}{2}$$

$$\therefore -EIw''(x) = \frac{M_b g x}{2} - m_o - \frac{M_b g x^2}{2L} \qquad (4)$$

Microstructures

- **Double-clamped beam (bridge):**



- Boundary Conditions are,

$$w(0) = 0, w'(0) = 0, w(L) = 0, w'(L) = 0 \quad (5)$$

- Solving (4) and (5),

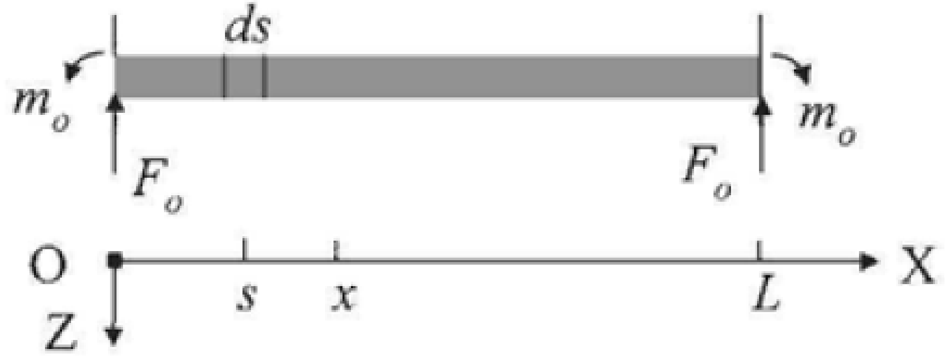
$$\therefore m_o = \frac{M_b g L}{12}$$

- The displacement function will be:

$$w(x) = \frac{M_b g}{24ELI} x^2 (L - x)^2 = \frac{\rho g}{2Eh^2} x^2 (L - x)^2$$

Microstructures

- **Double-clamped beam (bridge):**



- Maximum deflection is:

- Bridge stiffness is,

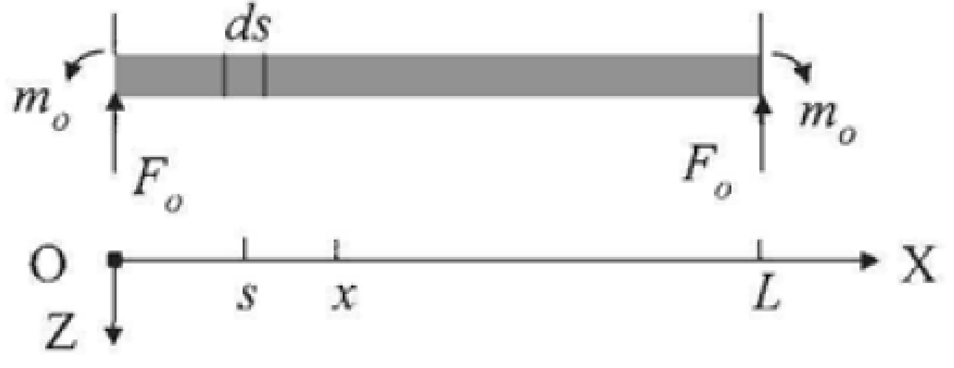
$$\therefore K = \frac{M_b g}{w_{\max}} = \frac{384EI}{L^3} = \frac{32Ebh^3}{L^3}$$

- The resonant frequency will be:

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K}{M_b}} = \frac{1}{2\pi}\sqrt{\frac{32Ebh^3}{L^3 M_b}}$$

Microstructures

- **Double-clamped beam (bridge):**



- Stress on the top of the beam (z = -h/2) is:

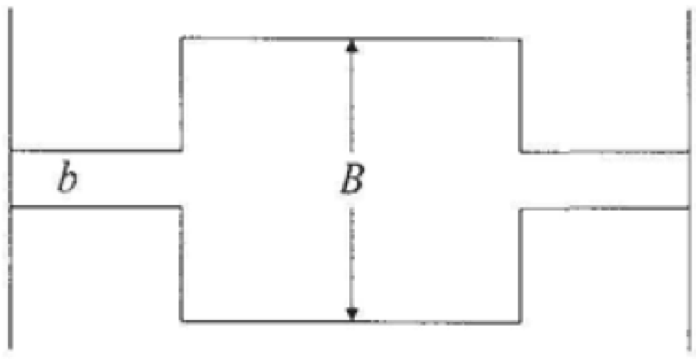
$$\therefore T(x) = -E\left(-\frac{h}{2}\right)w''(x) = \frac{M_b g h}{24LI}(L^2 - 6Lx + 6x^2)$$

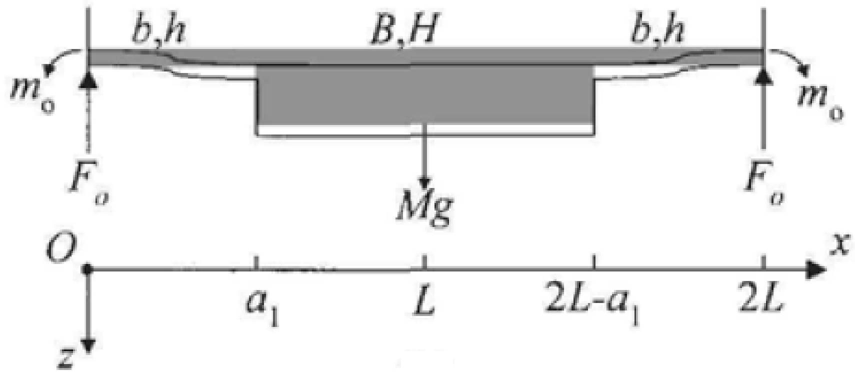
- Maximum stress on the top of the beam (*x = 0*) is:

$$\therefore T(x=0) = T_{\max} = \frac{M_b g h L^2}{24LI} = \frac{\rho g L^2}{2h}$$

Microstructures

Double-clamped beam with central mass:

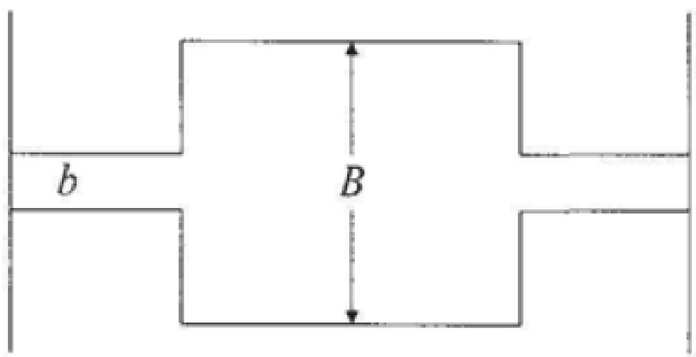


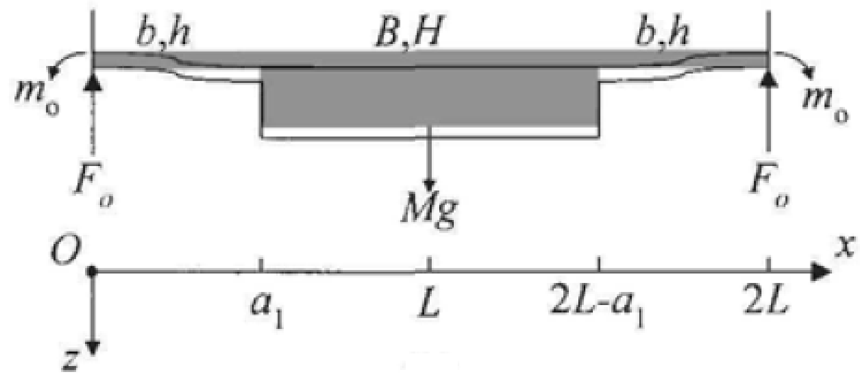


- Usually this structure is used for inertial sensors (such as accelerometers), resonators and switches.
- As shown in figure, the central mass is much wider and thicker than the beams, so we can ignore bending of the central mass.
- Due to symmetry, one half will be considered.

Microstructures

- **Double-clamped beam with central mass:**





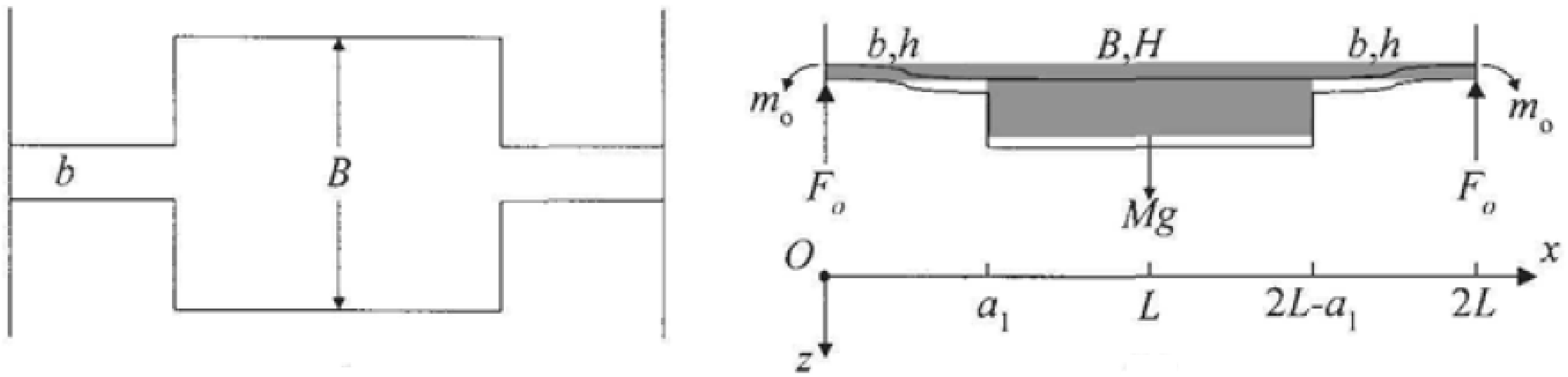
- At point x and from left hand side,

$$\therefore -EIw''(x) = F_o x - m_o - \frac{M_b g x^2}{2L}$$

$$\because F_o = \frac{(M + M_b)g}{2}, \quad M_b = 2\rho bhL, \quad M = 2\rho BH(L - a_1)$$

Microstructures

- **Double-clamped beam with central mass:**



- If $M_b << M$

$$\therefore -EIw''(x) = \frac{Mgx}{2} - m_o \qquad (7)$$

- Boundary Conditions are,

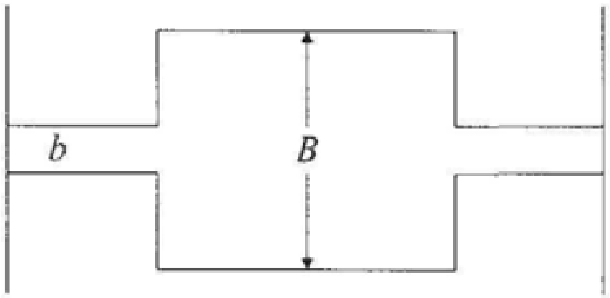
$$w(0)=0,\ w'(0)=0,\ w'(a_1)=0,\ w''(\frac{1}{2}a_1)=0 \qquad (8)$$

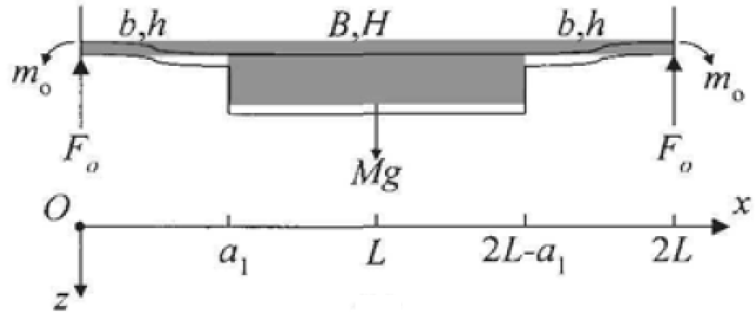
- Solving (7) and (8);

$$\therefore m_o = \frac{Mga_1}{4}$$

Microstructures

- **Double-clamped beam with central mass:**



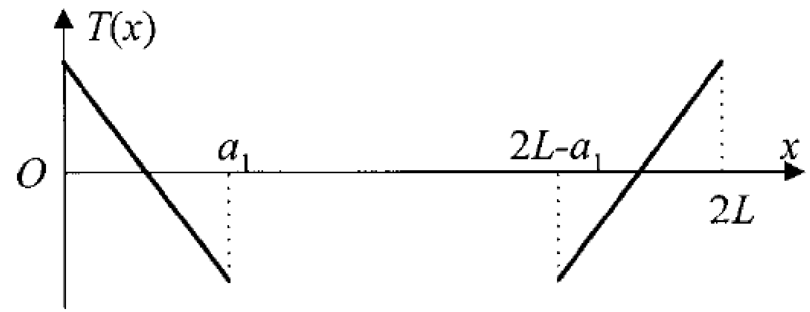


- The stress on the top of the beam is

$$\therefore T(x) = \frac{3Mg}{bh^2}\left(\frac{1}{2}a_1 - x\right)$$

- Maximum tensile stress will be @ *x = 0*,

$$\therefore T(x=0) = \frac{3Mga_1}{2bh^2}$$



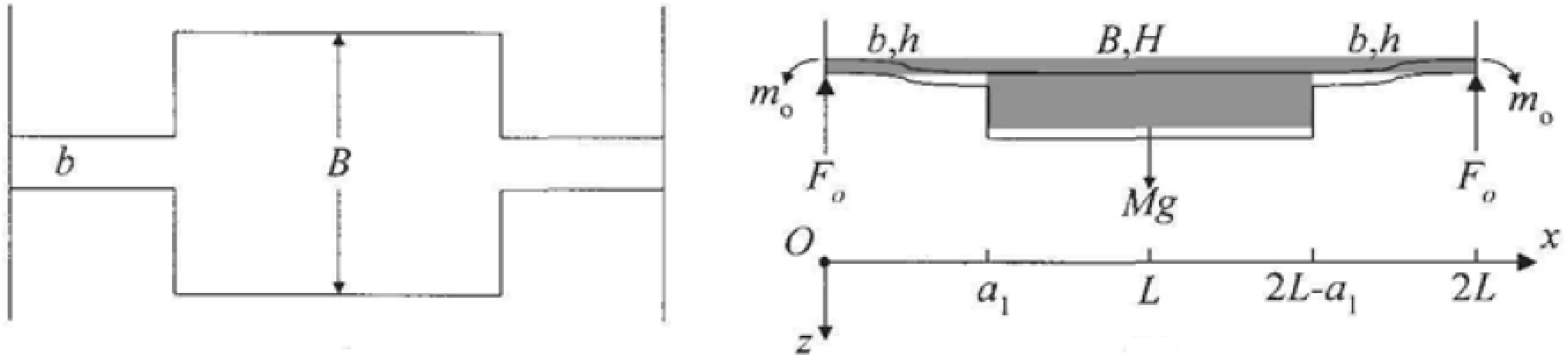
The stress distribution on the beam surface

- Maximum compressive stress will be @ *x* =

$$\therefore T(x=a_1) = -\frac{3Mga_1}{2bh^2}$$

Microstructures

- **Double-clamped beam with central mass:**



- The displacement function can be found as:

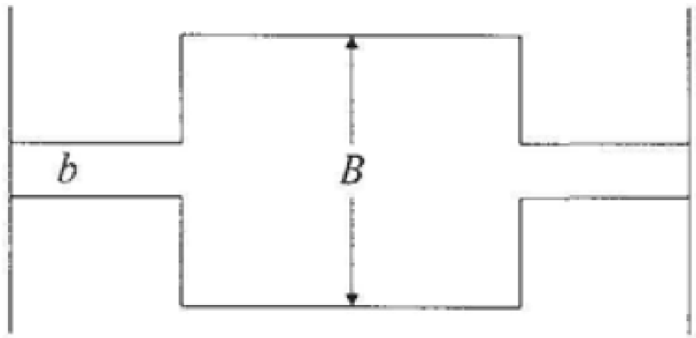
$$\therefore w(x) = \frac{Mg}{Ebh^3} x^2 \left(\frac{3}{2} a_1 - x \right)$$

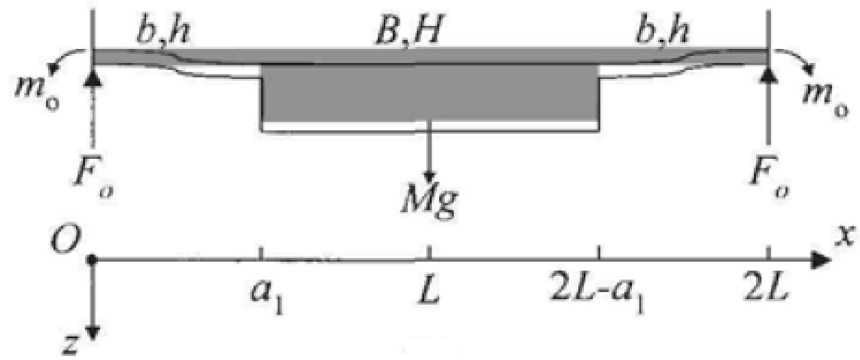
- Displacement of the central mass (maximum deflection), @ $w = a_1$,

$$\therefore w(x = a_1) = \frac{Mg}{2Ebh^3} a_1{}^3$$

Microstructures

- **Double-clamped beam with central mass:**





- Spring stiffness can be calculated:

$$\therefore K = \frac{Mg}{w_{\max}} = \frac{2Ebh^3}{a_1^{\ 3}}$$

- Resonant frequency:

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{2Ebh^3}{a_1^{\ 3}M}}$$

Can you find ***K*** for this structure using the stiffness calculated from the simple beam structure????

Beam Orientation

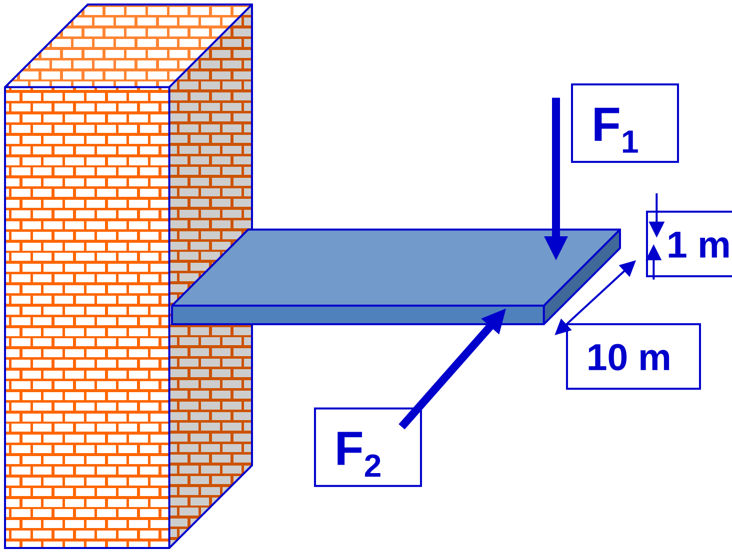
- Beam stiffness, beam spring constant, depends on the orientation of the beam.

$$K_1 = \frac{F_1}{w(L)} = \frac{F_1}{\left\{\frac{F_1 L^3}{3EI_1}\right\}}$$

- Where I_1 is the second moment of area perpendicular to the direction of force F_1.

$$I_1 = \frac{\text{beam width(beam hieght)}^3}{12} = \frac{(10)(1)^3}{12}$$

$$\therefore K_1 = \frac{F_1}{\left\{\frac{F_1 L^3}{3EI_1}\right\}} = \frac{3EI_1}{L^3} = \frac{3E\frac{(10)(1)^3}{12}}{L^3} = \frac{3E(10)(1)^3}{12L^3}$$



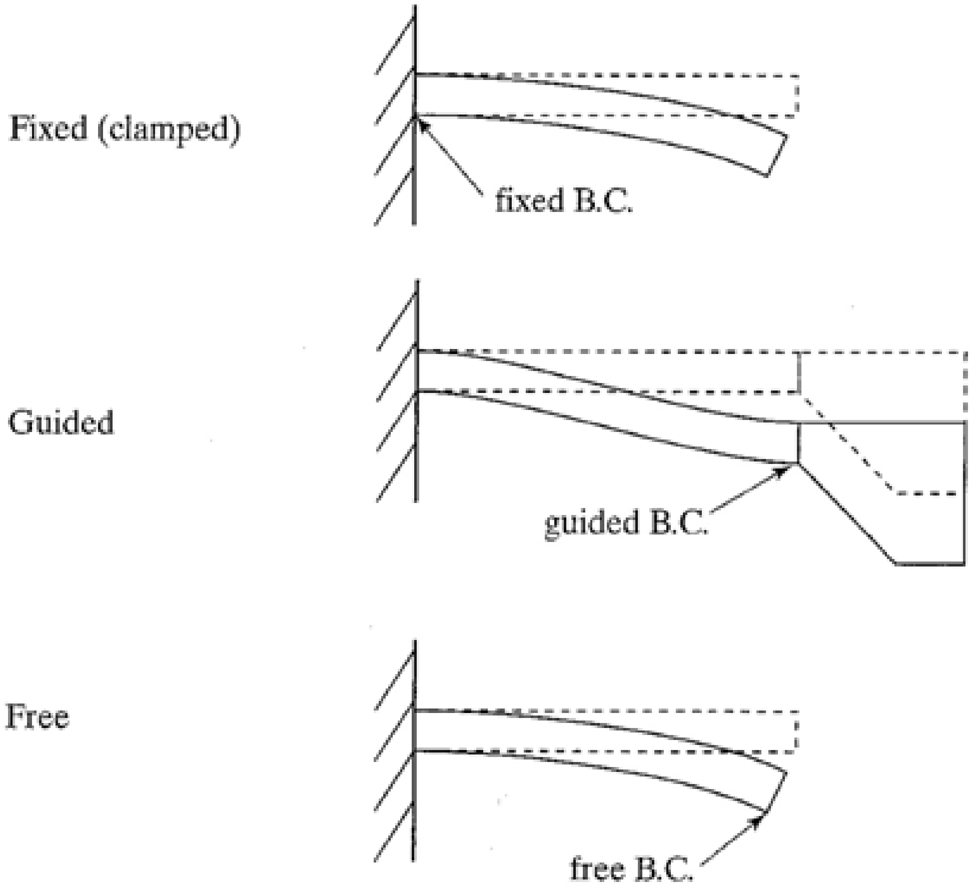
$$\therefore K_2 = \frac{F_2}{\left\{\frac{F_2 L^3}{3EI_2}\right\}} = \frac{3EI_2}{L^3} = \frac{3E(1)(10)^3}{12L^3}$$

To get K_2, we need to calculate I_2

$$I_2 = \frac{\text{beam hieght(beam width)}^3}{12} = \frac{(1)(10)^3}{12}$$

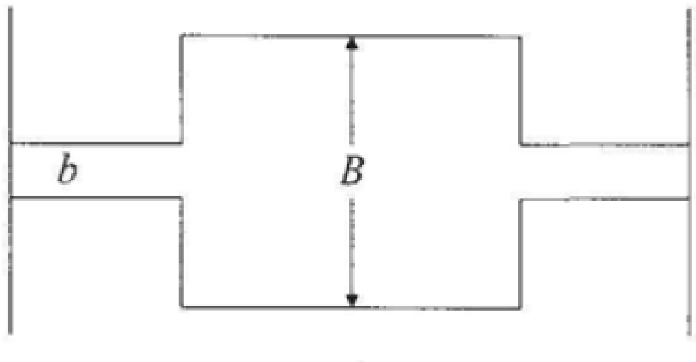
$$\therefore K_2 = 100K_1$$

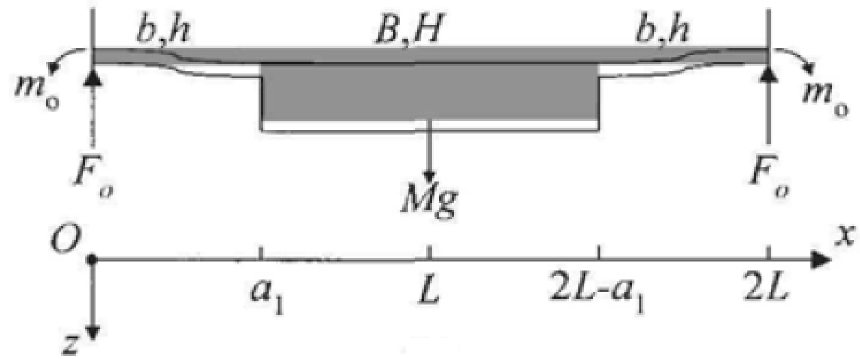
Beam Boundary Conditions



Reminder with previous Q.

- **Double-clamped beam with central mass:**



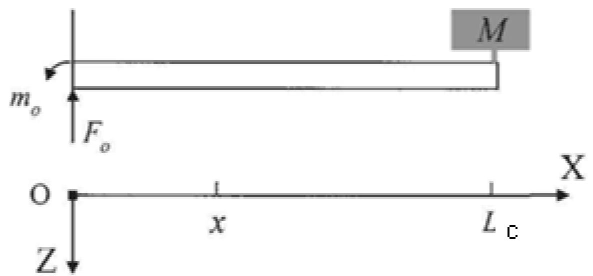


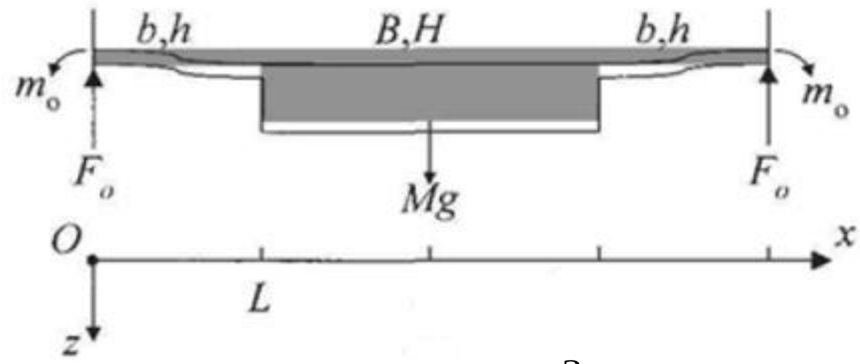
- Spring stiffness can be calculated:

$$\therefore K = \frac{Mg}{w_{max}} = \frac{2Ebh^3}{a_1^{\ 3}}$$

Can you find ***K*** for this structure using the stiffness calculated from the simple beam structure????

Beam Combinations



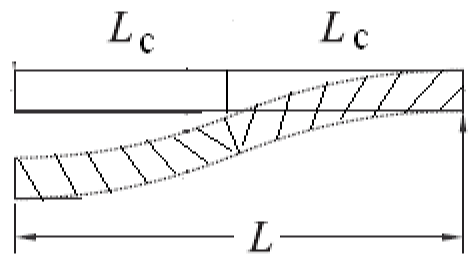


$$K_c = \frac{Ebh^3}{4L_c^{\ 3}}$$

$$K_{gc} = \frac{2Ebh^3}{L^3}$$

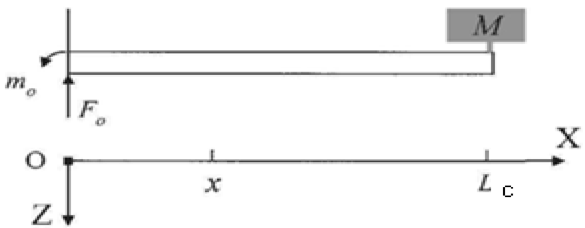
Guided end beam can be considered as 2 free end beams connected in series

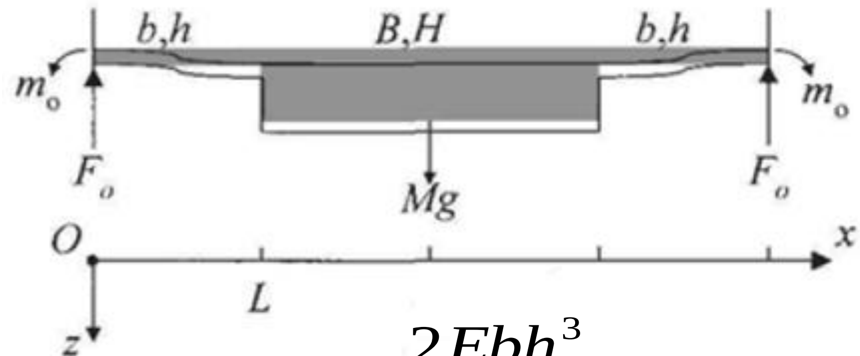
Fixed-guided beam



$$L_c = \frac{1}{2}L$$

Beam Combinations

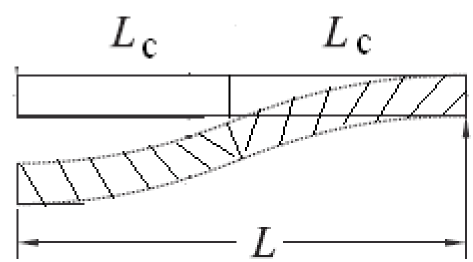




$$K_c = \frac{Ebh^3}{4L_c^{\ 3}}$$

$$K_{gb} = K_{gb(left)} \,//\, K_{gb(right)}$$

$$K_{gb} = \frac{2Ebh^3}{L^3}$$



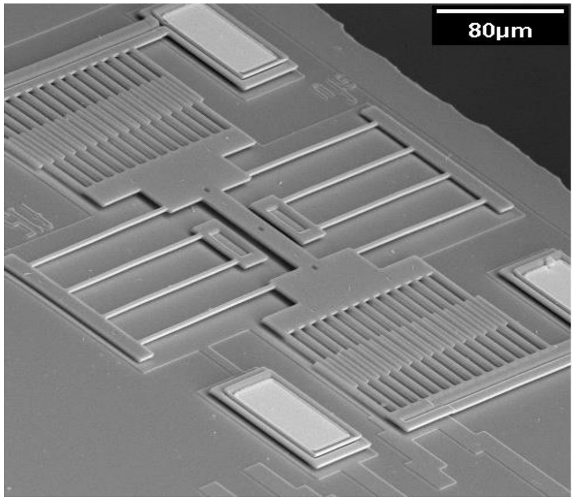
$$K_{gb} = 2K_{gb(left)} = 2(K_c + K_c) = 2\frac{K_c}{2} = K_c$$

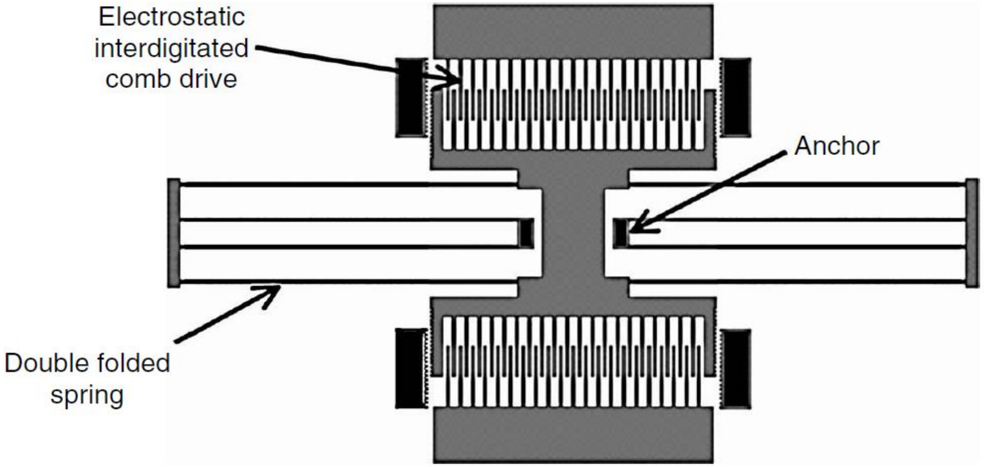
PROOF

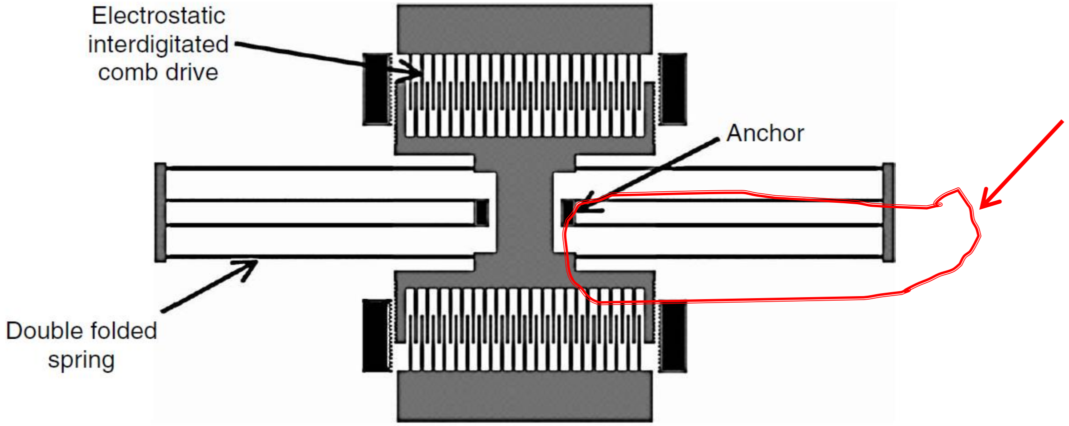
$$K_{gb} = \frac{2Ebh^3}{(2L_c)^3} = \frac{2Ebh^3}{8L_c^{\ 3}} = \frac{Ebh^3}{4L_c^{\ 3}}$$

This structure eliminates the rotation of the mass, but increases the strain (elongation) of the beams and increases the stress in beams in return.

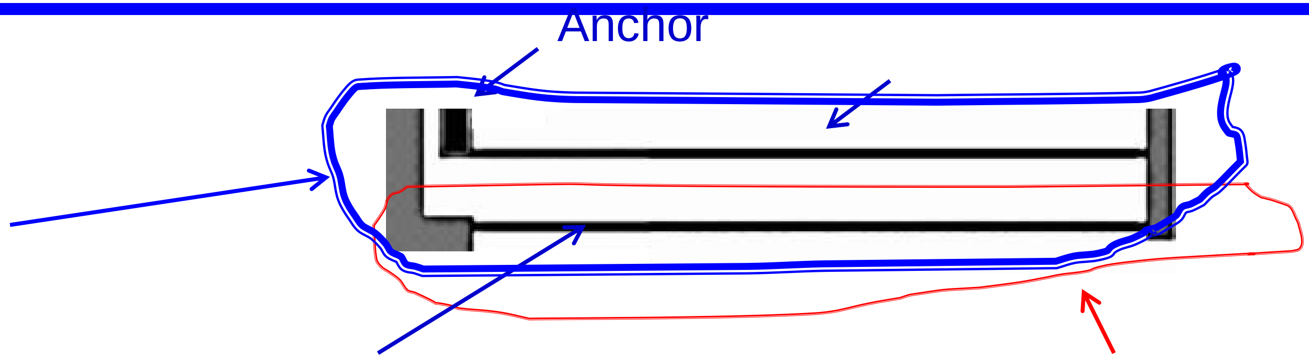
Folded Beam Springs





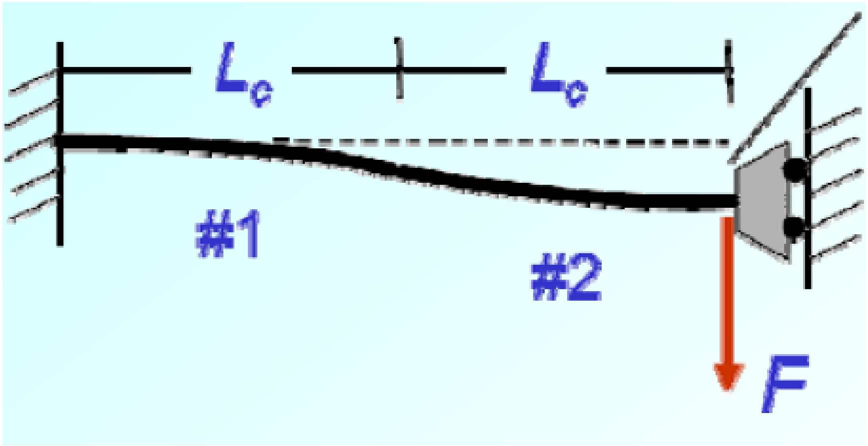


Folded Beam Springs

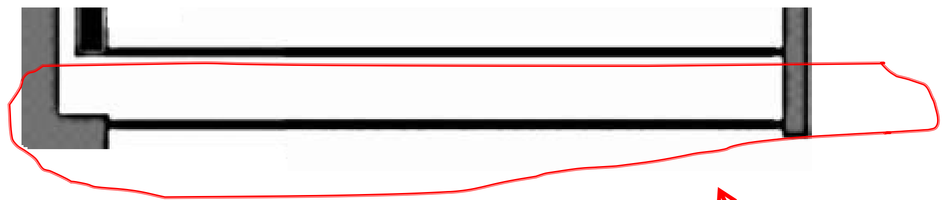


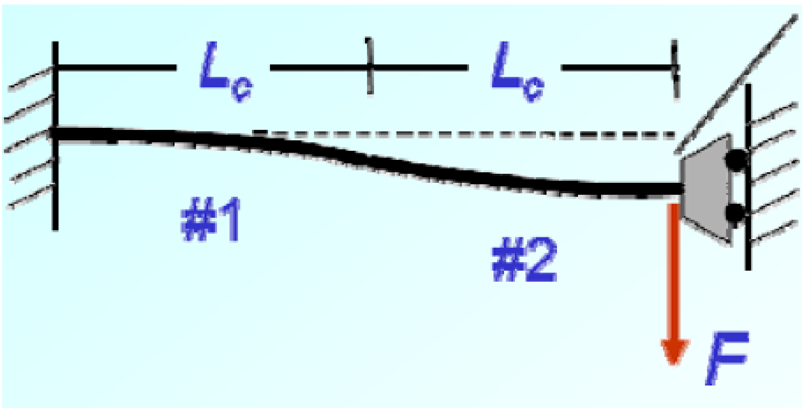
$$K_F = K_{gb(upper)} + K_{gb(lower)}$$

$$K_F = \frac{1}{2} K_{gb(lower)}$$



Folded Beam Springs





$$K_F = K_{gb(upper)} + K_{gb(lower)}$$

$$K_F = \frac{1}{2} K_{gb(lower)} = \frac{Ebh^3}{16 L_c^3}$$

$$K_{gb(lower)} = \frac{1}{2} K_c = \frac{1}{2} \frac{Ebh^3}{4 L_c^3} = \frac{Ebh^3}{8 L_c^3}$$

$$K_S = 4(K_{gb(lower)}) = \frac{Ebh^3}{4 L_c^3}$$

Discussion

- **<u>For fixed-guided beam suspensions:</u>**
  - **The spring is linear for only small deflections, however, for large deflections the spring is nonlinear.**
  - **This is due to the high axial forces induced in the beams.**
  - **We should avoid such kind of suspensions in case of mechanisms require large deflections..**
- **<u>For folded beam suspensions:</u>**
  - **This suspension eliminates the high axial force produced in the fixed-guided beams, and hence reduces the nonlinearity.**
  - **Is considered as a highly linear spring.**