Faculty of Engineering

CSE115: Digital Design

Lecture 5:
Boolean Switching Algebra

Suggested Reading

- Section 4.1

Boolean ('Switching') Algebra

- **Boolean values**: 0, 1
- **Positive-logic convention**: analog voltages LOW, HIGH → 0, 1
- **Signal values**: denoted by variables (**X, Y, FRANK,** etc.)
- **Complement**: $\mathbf{X'}$ (opposite of **X**)
- **AND**: $\mathbf{X \cdot Y}$
- **OR**: $\mathbf{X + Y}$
- **Literal**: a variable or its complement: **X, X′, FRANK′**
- **Expression**: literals combined by AND, OR, parentheses

$$(A \cdot B' \cdot C + Q5) \cdot RESET'$$

- **Equation**: Variable = expression

$$P = ((FRANK \cdot Z') + (A \cdot B' \cdot C + Q5) \cdot RESET'$$

Basic Axioms

- A variable can take only one of two values (**0,1**)
  (A1) $X = 0 \text{ if } X \neq 1$ (A1') $X = 1 \text{ if } X \neq 0$

- **NOT** operation (The complement Operation):
  (A2) $\text{If } X = 0 \text{ then } X \neq 1$ (A2') $\text{If } X = 1 \text{ then } X \neq 0$

- **AND** and **OR** operations (Multiplication and Addition):
  (A3) $0 \cdot 0 = 0$ (A3') $0 + 0 = 0$
  (A4) $1 \cdot 1 = 1$ (A4') $1 + 1 = 1$
  (A5) $0 \cdot 1 = 1 \cdot 0 = 0$ (A5') $1 + 0 = 0 + 1 = 1$

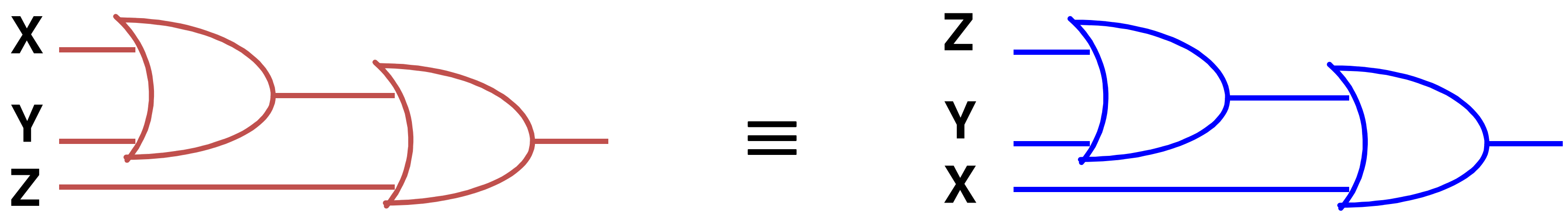
Theorems - Single Variable

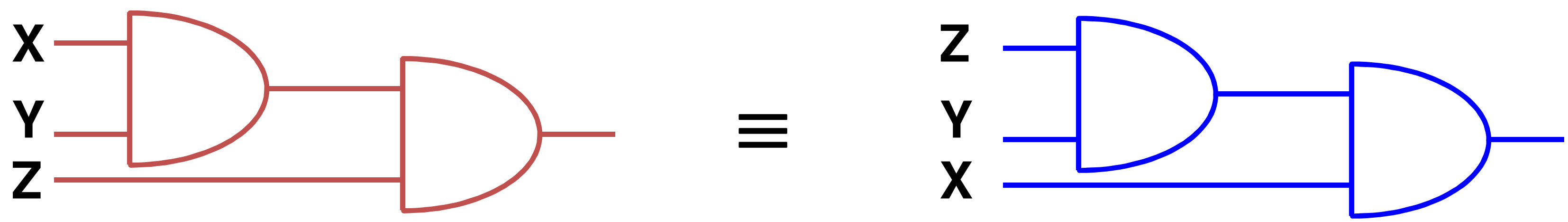
- **Identity elements**: (T1) $X + 0 = X$ (T1′) $X \cdot 1 = X$
- **Null elements**: (T2) $X + 1 = 1$ (T2′) $X \cdot 0 = 0$
- **Idempotency**: (T3) $X + X = X$ (T3′) $X \cdot X = X$
- **Involution**: (T4) $(X')' = X$
- **Complements**: (T5) $X + X' = 1$ (T5′) $X \cdot X' = 0$
- **Induction Proof**:

  Show that the theorems are true for both $X = 0$ and $X = 1$

Theorems-Multiple Variables

- **Commutativity:** (T6) $X + Y = Y + X$ (T6') $X \cdot Y = Y \cdot X$
  - The inputs of AND and OR gates can be interchanged.
- **Associativity:** (T7) $(X + Y) + Z = X + (Y + Z)$ (T7') $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$
  - The order of the input variables could be rearranged.





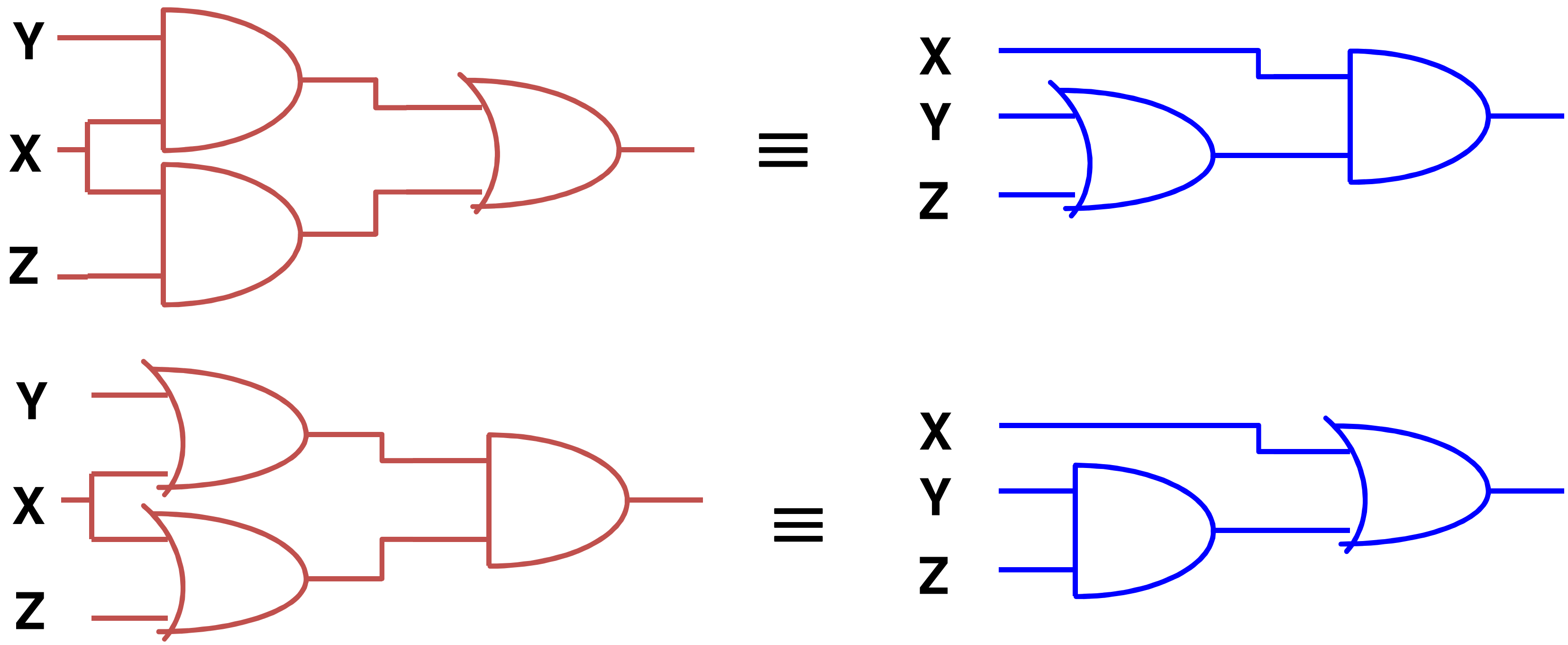
Theorems - Multiple Variables

- **Distributivity:**

  (T8) $\mathbf{X \cdot (Y + Z) = X \cdot Y + X \cdot Z}$

  (T8') $\mathbf{X + Y \cdot Z = (X + Y) \cdot (X + Z)}$

  - Multiplication distributes over addition; Addition distributes over multiplication!



Theorems-Multiple Variables

- **Covering:** (T9) $\mathbf{X + X \cdot Y = X}$ (T9') $\mathbf{X \cdot (X + Y) = X}$

Proof:

(T9): $\mathbf{X + X \cdot Y = X \cdot 1 + X \cdot Y}$ (theorem T1')

$= \mathbf{X \cdot (1 + Y)}$ (theorem T8- Distributivity)

$= \mathbf{X \cdot 1}$ (theorem T2)

$= \mathbf{X}$ (theorem T1')

(T9'): $\mathbf{X \cdot (X + Y) = (X + 0) \cdot (X + Y)}$ (theorem T1)

$= \mathbf{X + (0 \cdot Y)}$ (theorem T8'- Distributivity)

$= \mathbf{X + 0}$ (theorem T2')

$= \mathbf{X}$ (theorem T1)

Theorems-Multiple Variables

- **Combining:** (T10) $X \cdot Y + X \cdot Y' = X$ (T10') $(X + Y) \cdot (X + Y') = X$

Proof:

(T10)	$X = X \cdot 1$	(theorem T1')
	$= X \cdot (Y + Y')$	(theorem T5)
	$= X \cdot Y + X \cdot Y'$	(theorem T8 - Distributivity)
(T10')	$X = X + 0$	(theorem T1)
	$= X + (Y \cdot Y')$	(theorem T5')
	$= (X + Y) \cdot (X + Y')$	(theorem T8' - Distributivity)

Covering and combining are used in minimizing logic functions.

Theorems-Multiple Variables

- **Consensus:**

(T11) $\mathbf{X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z}$

(T11’) $\mathbf{(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)}$

- **Generalized Idempotency:**

(T12) $\mathbf{X + X + \ldots + X = X}$

(T12’) $\mathbf{X \cdot X \cdot \ldots \cdot X = X}$

Theorems-Multiple Variables

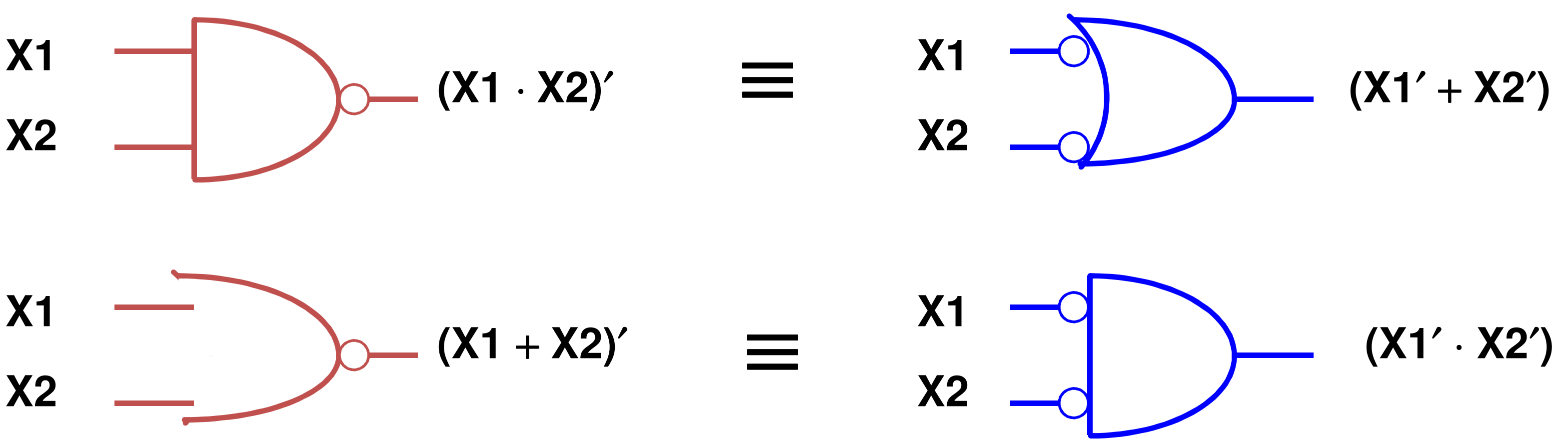
MOST IMPORTANT!!!

- **DeMorgans Theorems**

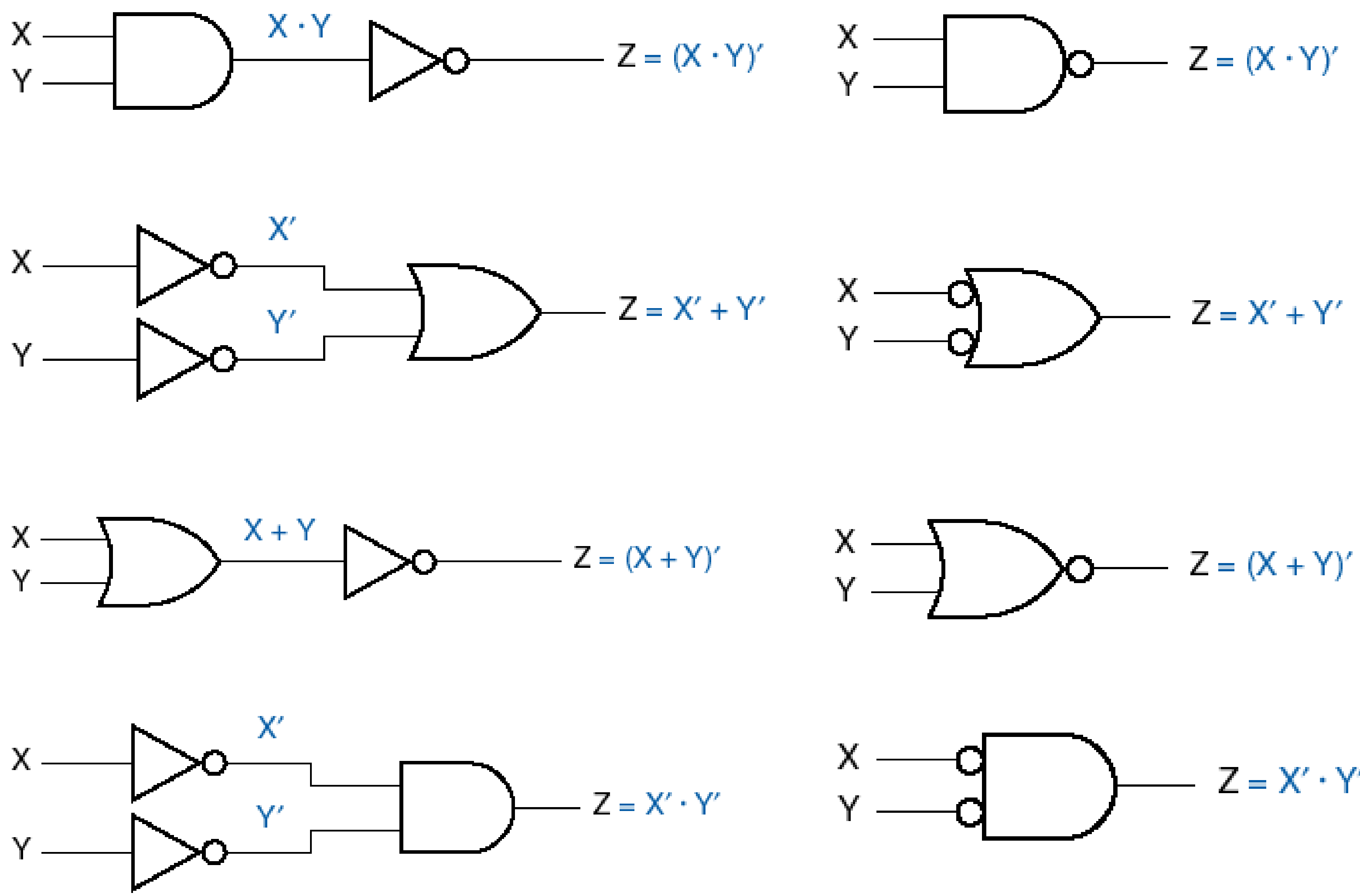
(T13) $(X1 \cdot X2 \cdot \ldots \cdot Xn)' = X1' + X2' + \ldots + Xn'$

(T13') $(X1 + X2 + \ldots + Xn)' = X1' \cdot X2' \cdot \ldots \cdot Xn'$

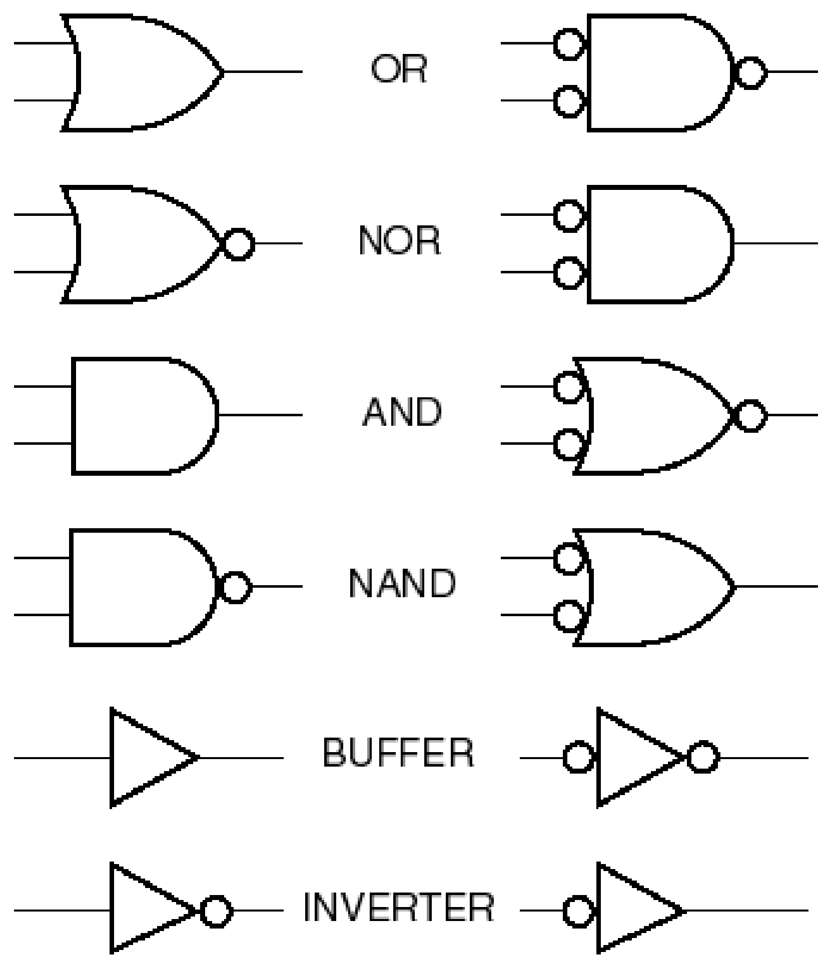
Example: two-variable case



DeMorgan Symbol Equivalence



DeMorgan Symbols

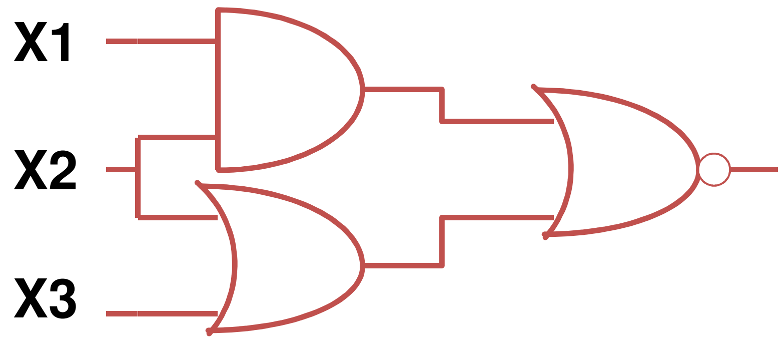


Generalized DeMorgan's Theorem

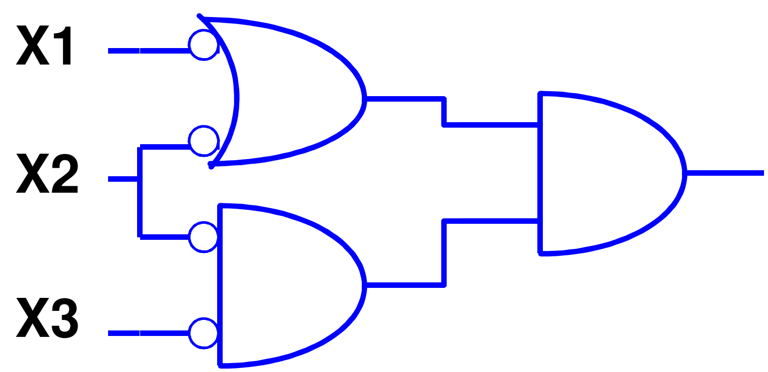
(T14) $[F(X1, X2, \ldots, Xn, +, \cdot)]' = F(X1', X2', \ldots, Xn', \cdot, +)$

Example: $F = (X1 \times X2) + (X2 + X3)$

$F' = [(X1 \times X2) + (X2 + X3)]'$



$F' = (X1' + X2') \times (X2' \times X3')$



Duality

- **Swap 0 & 1, AND & OR**
  - **Result: Theorems still true**
- Why?
  - Each axiom (A1-A5) has a dual (A1’-A5’)
- Counterexample:

  $X + X \cdot Y = X$ (T9)

  $X \cdot X + Y = X$ (dual)

- **Mathematical definition :** F is a Boolean Function; FD the dual function is:
  - $\mathbf{FD}\ (\mathbf{X1},\mathbf{X2},\ \ldots\ ,\mathbf{Xn},\ +,\ \cdot\ ,\ ')\ =\ \mathbf{F}\ (\mathbf{X1},\mathbf{X2},\ \ldots\ ,\mathbf{Xn},\ \cdot\ ,\ +,\ ')$

Duality

Example:

- $F_{X1,X2,X3} = X1 + X2 \cdot X3$
- $FD_{X1,X2,X3} = X1 \cdot (X2 + X3)$
- $FD'_{X1,X2,X3} = (X1 \cdot (X2 + X3))' = X1' + X2' \cdot X3'$

  $= F_{X1', X2', X3'}$
- $FD_{X1',X2',X3'} = X1' \cdot (X2' + X3')$
- $FD'_{X1',X2',X3'} = (X1' \cdot (X2' + X3')' = X1'' + X2'' \cdot X3''$

  $= X1 + X2 \cdot X3$

$$F_{X1,X2,X3} = FD'_{X1',X2',X3'}$$

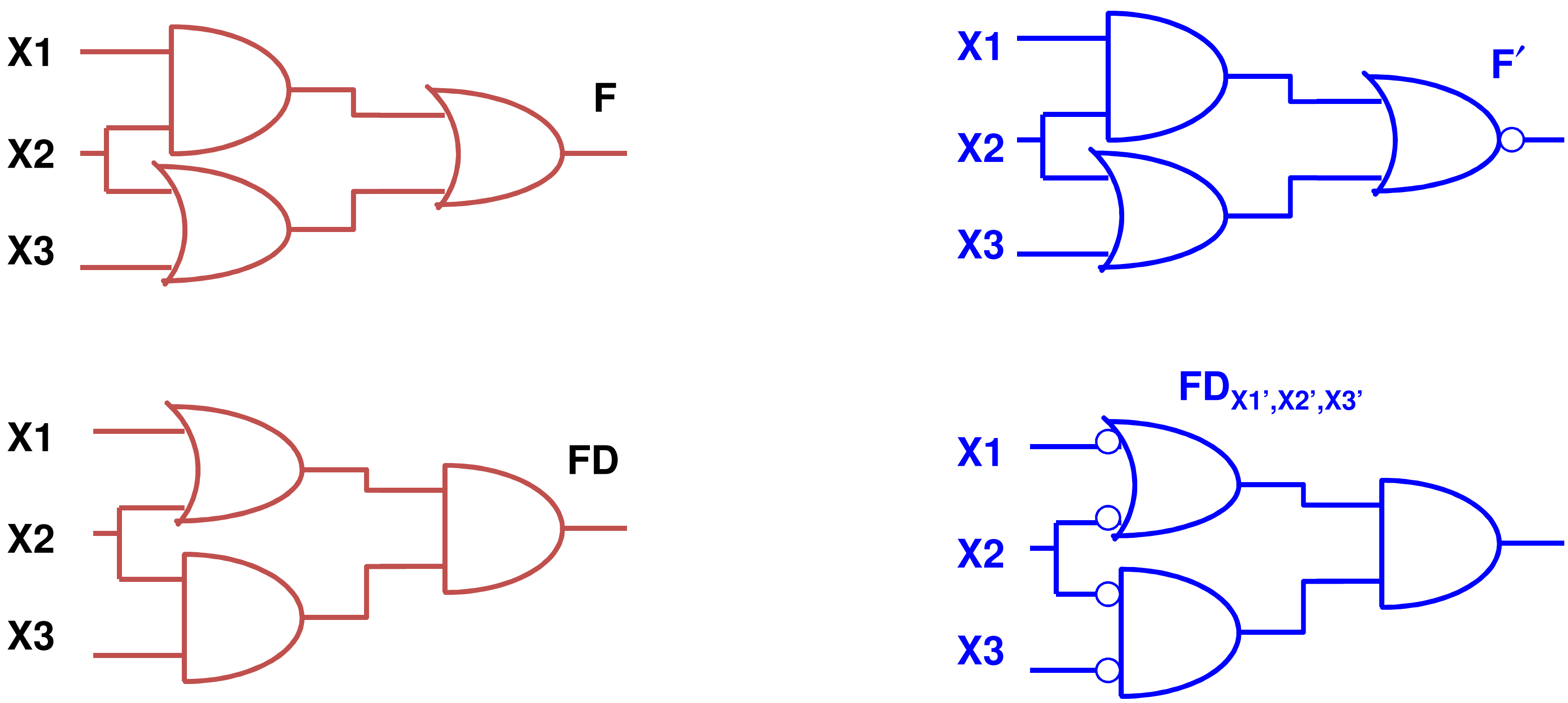
Duality and DeMorgan's Theorem

$$F_{X1,X2,\ldots,Xn} = FD'_{X1',X2',\ldots,Xn'}$$

or:

$$F'_{X1,X2,\ldots,Xn} = FD_{X1',X2',\ldots,Xn'}$$

Example:



Exercise: DeMorgan/Duality

Find: F', FD, $FD_{A',B',C'}$, $FD'_{A',B',C'}$

$$F_{A,B,C} = AB + AB'C + BC'$$

$$F' = [A \times B + A \times B' \times C + B \times C']'$$

$$= (A' + B') \times (A' + B + C') \times (B' + C)$$

$$FD = (A + B) \times (A + B' + C) \times (B + C')$$

$$FD_{A',B',C'} = (A' + B') \times (A' + B + C') \times (B' + C)$$

$$FD'_{A',B',C'} = [(A' + B') \times (A' + B + C') \times (B' + C)]'$$

$$= A \times B + A \times B' \times C + B \times C'$$

Representation of Logic Functions

- **Truth table** with 2^n rows, n: the number of variables
- **Literal**: a variable or its complement
  - X, Y'
- n-variable **minterm**: *product* term with n literals
  - $X' \cdot Y \cdot Z$
- n-variable **maxterm**: *sum* term with n literals
  - $X + Y' + Z$

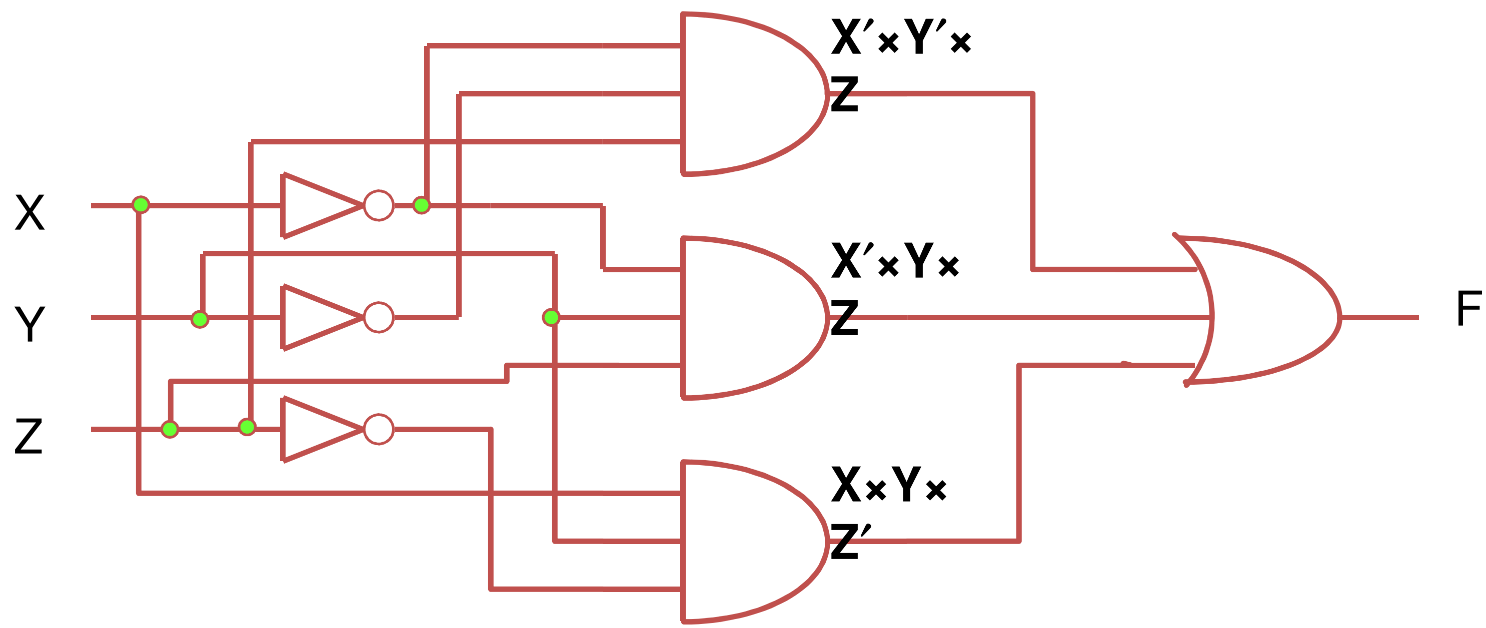
Canonical Representation

- **Canonical Sum** (Sum of Products - SOP):
  - Sum of minterms corresponding to input combinations for which the function produces a 1 output.
  - $\mathbf{F_{X,Y,Z} = \Sigma\ (1,3,6)}$
  - $\mathbf{F = X' \cdot Y' \cdot Z + X' \cdot Y \cdot Z + X \cdot Y \cdot Z'}$

- **Canonical Product** (Product of Sums - POS):
  - Product of maxterms corresponding to input combinations for which the function produces a 0 output.
  - $\mathbf{F_{X,Y,Z} = \Pi\ (0,2,4,5,7)}$
  - $\mathbf{F = (X + Y + Z) \cdot (X + Y' + Z) \cdot (X' + Y + Z) \cdot (X' + Y + Z') \cdot (X' + Y' + Z')}$

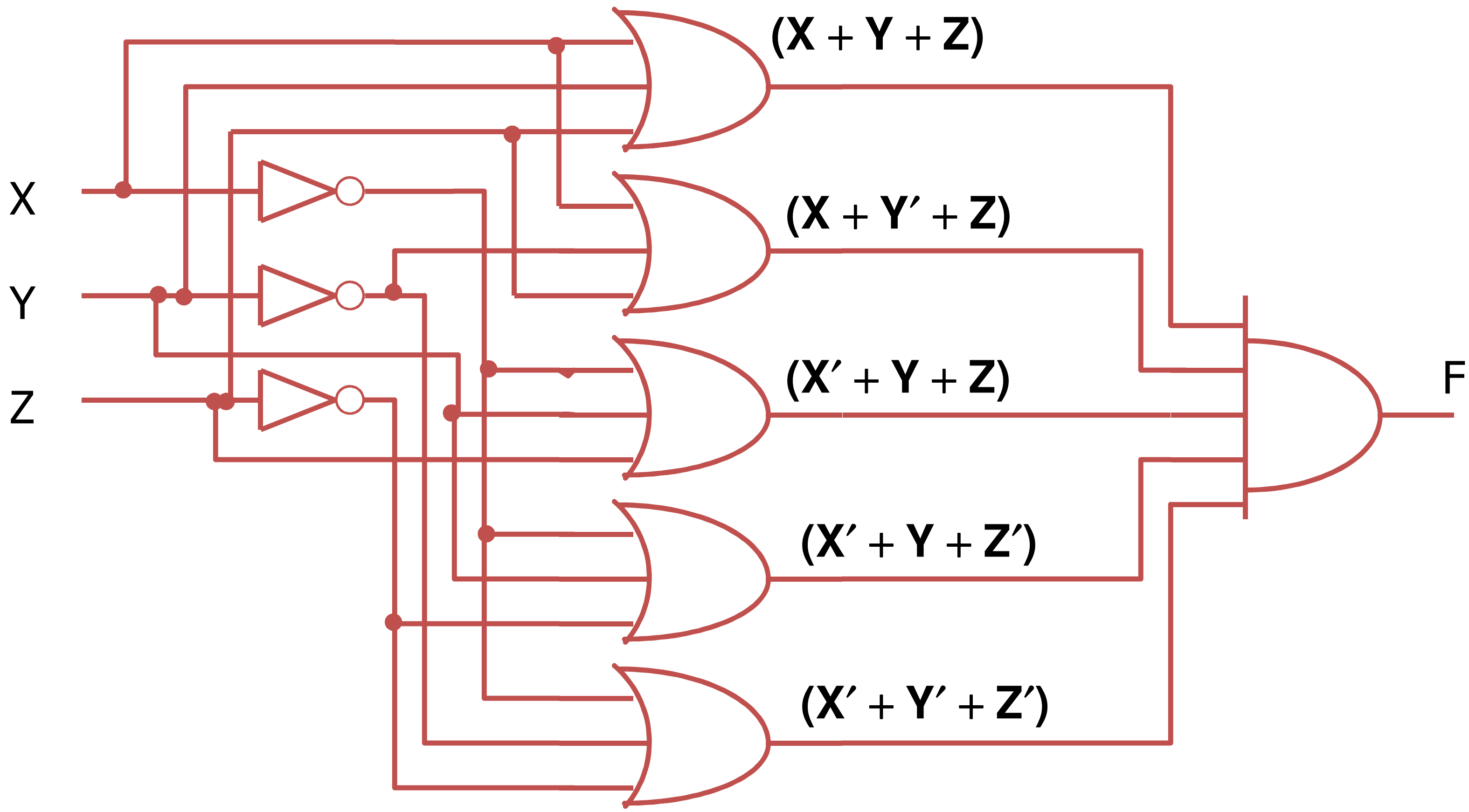
Canonical Sum Implementation

$$F = X' \times Y' \times Z + X' \times Y \times Z + X \times Y \times Z'$$



Canonical Product Implementation

$$F = (X + Y + Z) \times (X + Y' + Z) \times (X' + Y + Z) \times (X' + Y + Z') \times (X' + Y' + Z')$$



Logic Function Simplification

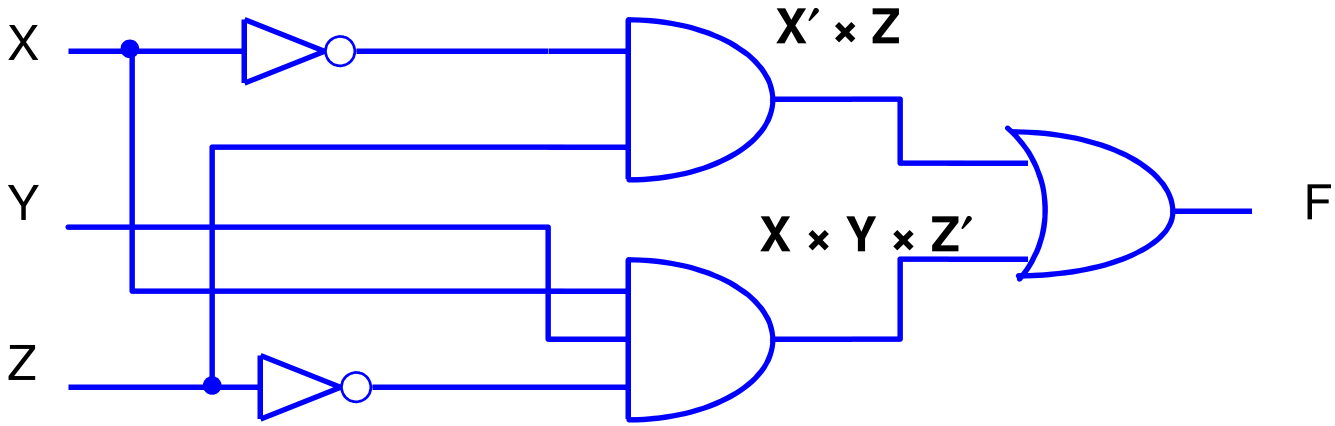
$$F = X' \times Y' \times Z + X' \times Y \times Z + X \times Y \times Z'$$

$$= X' (Y' \times Z + Y \times Z) + X \times Y \times Z'$$

$$= X' ((Y' + Y) \times Z) + X \times Y \times Z'$$

$$= X' (1 \times Z) + X \times Y \times Z'$$

$$= X' \times Z + X \times Y \times Z'$$



Exercise

F(A,B,C) = Σ (0,2,4,7); Write:

The Truth Table; The Canonical Sum (SOP); The Canonical Product (POS)

Row	X	Y	Z	F	Minterms	Maxterms
0	0	0	0			
1	0	0	1			
2	0	1	0			
3	0	1	1			
4	1	0	0			
5	1	0	1			
6	1	1	0			
7	1	1	1			

SOP: $F = A' \times B' \times C' + A' \times B \times C' + A \times B' \times C' + A \times B \times C$

POS: $F = (A + B + C') \times (A + B' + C') \times (A' + B + C') \times (A' + B' + C)$