Problem 1.

Using the discrete Fourier transform:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-\frac{j2\pi ux}{M}}$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} e^{-\frac{j2\pi ux}{M}}$$

Using the expansion of the geometric series:

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

In the previous equation

$$F(u) = \frac{1}{M}\frac{1-\left(e^{-\frac{j2\pi u}{M}}\right)^M}{1-e^{-\frac{j2\pi u}{M}}}$$

For the numerator since 1 = $e^{-j\pi u}e^{j\pi u}$

$$F(u) = \frac{1}{M}\frac{e^{-j\pi u}e^{j\pi u} - e^{-j2\pi u}}{1-e^{-\frac{j2\pi u}{M}}}$$

$$F(\omega) = \frac{1}{M}\frac{e^{-j\pi u}(e^{j\pi u} - e^{-j\pi u})}{1-e^{-\frac{j2\pi u}{M}}}$$

Similarly for the numerator:

$$F(\omega) = \frac{1}{M}\frac{e^{-j\pi u}\left(e^{j\pi u} - e^{-j2\pi u}\right)}{e^{-\frac{j\pi u}{M}}(e^{j\pi u} - e^{-j\pi u})}$$

Using Euler's formula $e^{j\theta} = \cos(\theta) + jsin(\theta)$ to simplify the numerator and the denominator

$$F(u) = \frac{1}{M}e^{-j\pi u\left(1-\frac{1}{M}\right)}\left(\frac{\sin(\pi u)}{\sin\left(\frac{\pi u}{M}\right)}\right) \qquad (1)$$

Using the sinc function defined as

$$sinc(x) = \frac{\sin(x)}{x}$$

Multiplying (1) by $\frac{\pi u}{\pi u}$

$$F(u) = e^{-j\pi u\left(1-\frac{1}{M}\right)}\frac{\left(\frac{\pi u}{M}\right)}{\sin\left(\frac{\pi u}{M}\right)}\frac{\sin(\pi u)}{\pi u}$$

$$F(u) = e^{-j\pi u\left(1-\frac{1}{M}\right)}\left(\frac{sinc(\pi u)}{\mathrm{sinc}\left(\frac{\pi u}{M}\right)}\right)$$

To draw $F(u)$ we need to get the amplitude

$$|F(u)| = \frac{\mathrm{sinc}(\pi u)}{\mathrm{sinc}\left(\frac{\pi u}{M}\right)}$$

Since u is always smaller than M ,The denominator will never be zero except at $u = 0$

At $u = 0$

$$|F(0)| = \frac{0}{0}\ (Undefined)$$

Taking the limit as $u \to 0$ and using l'hopital's rule

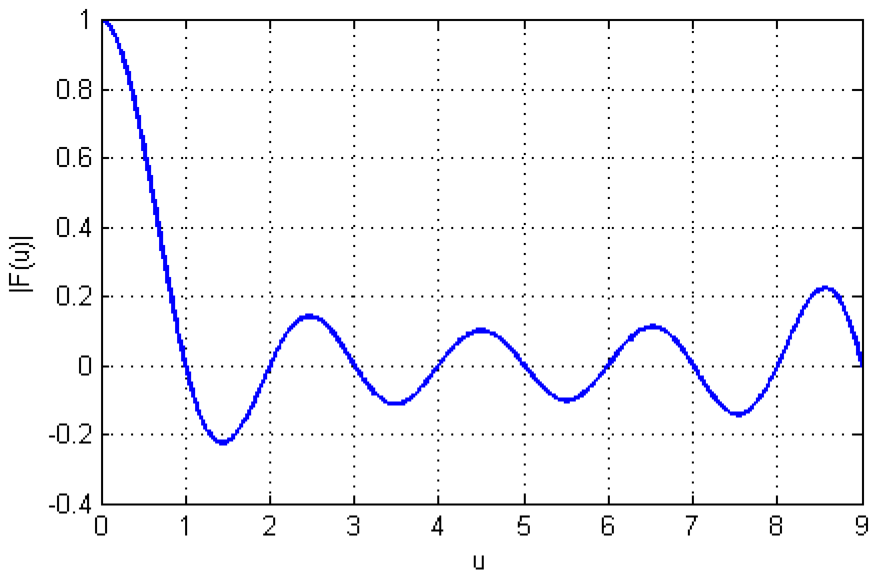
$$|F(0)| = \lim_{u\to 0}\frac{cos(\pi u)}{\cos\left(\frac{\pi u}{M}\right)} = 1$$

The numerator will be zero at $\pi u = \pi, 2\pi, 3\pi, \ldots.$

This should be only a schematic of $|F(u)|$, the most important part is that $F(0) = 1$

and $F(1), F(2), F(3), \ldots = 0$

For example for M = 10



Problem 2.

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} A\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \frac{A}{MN}\sum_{x=0}^{M-1} e^{-j2\pi\left(\frac{ux}{M}\right))}\sum_{y=0}^{N-1} e^{-j2\pi\left(\frac{vy}{N}\right))} \qquad (1)$$

Similar to Problem 1

$$\frac{1}{N}\sum_{y=0}^{N-1} e^{-j2\pi(\frac{vy}{N}))} = e^{-j\pi v\left(1-\frac{1}{N}\right)}\left(\frac{sinc(\pi v)}{\mathrm{sinc}\left(\frac{\pi \mathrm{v}}{\mathrm{N}}\right)}\right)$$

$$\frac{1}{M}\sum_{x=0}^{M-1} e^{-j2\pi(\frac{ux}{M}))} = e^{-j\pi u\left(1-\frac{1}{M}\right)}\left(\frac{sinc(\pi u)}{\mathrm{sinc}\left(\frac{\pi \mathrm{u}}{\mathrm{M}}\right)}\right)$$

In (1)

$$F(u,v) = A\, e^{-j\pi u\left(1-\frac{1}{M}\right)}\left(\frac{sinc(\pi u)}{\mathrm{sinc}\left(\frac{\pi \mathrm{u}}{\mathrm{M}}\right)}\right) e^{-j\pi v\left(1-\frac{1}{N}\right)}\left(\frac{sinc(\pi v)}{\mathrm{sinc}\left(\frac{\pi \mathrm{v}}{\mathrm{N}}\right)}\right)$$

Problem 3

Using the discrete Fourier transform:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-\frac{j2\pi ux}{M}}$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} (-1)^x e^{-\frac{j2\pi ux}{M}}$$

Since $-1 = e^{j\pi}$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} e^{j\pi x} e^{-\frac{j2\pi ux}{M}}$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} e^{-\frac{j2\pi ux}{M}+j\pi x}$$

$$\frac{1}{M}\sum_{x=0}^{M-1} e^{-\frac{j2\pi}{M}\left(u-\frac{M}{2}\right)x}$$

Using the expansion of the geometric series:

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

In the previous equation

$$F(u) = \frac{1}{M}\frac{1-\left(e^{-\frac{j2\pi}{M}\left(u-\frac{M}{2}\right)}\right)^M}{1-e^{-\frac{j2\pi}{M}\left(u-\frac{M}{2}\right)}}$$

To simplify things let $u-\frac{M}{2}=\theta$

$$F(u) = \frac{1}{M}\frac{1-\left(e^{-\frac{j2\pi\theta}{M}}\right)^{M}}{1-e^{-\frac{j2\pi\theta}{M}}}$$

This is similar to the formula in Problem 1 but with θ instead of u, using the same procedure

$$F(u) = e^{-j\pi\theta\left(1-\frac{1}{M}\right)}\left(\frac{sinc(\pi\theta)}{\mathrm{sinc}\left(\frac{\pi\theta}{M}\right)}\right)$$

To draw $F(u)$ we need to get the amplitude

$$|F(u)| = \frac{\mathrm{sinc}(\pi\theta)}{\mathrm{sinc}\left(\frac{\pi\theta}{M}\right)}$$

Substituting by $\theta = u - M/2$

$$|F(u)| = \frac{\mathrm{sinc}\left(\pi u - \frac{\pi M}{2}\right)}{\mathrm{sinc}\left(\frac{\pi u}{M} - \frac{\pi}{2}\right)}$$

The denominator well be zero $\frac{\pi u}{M} - \frac{\pi}{2} = 0 \quad \rightarrow \qquad u = \frac{M}{2}$

At $u = \frac{M}{2}$

$$|F(0)| = \frac{0}{0}\ (Undefined)$$

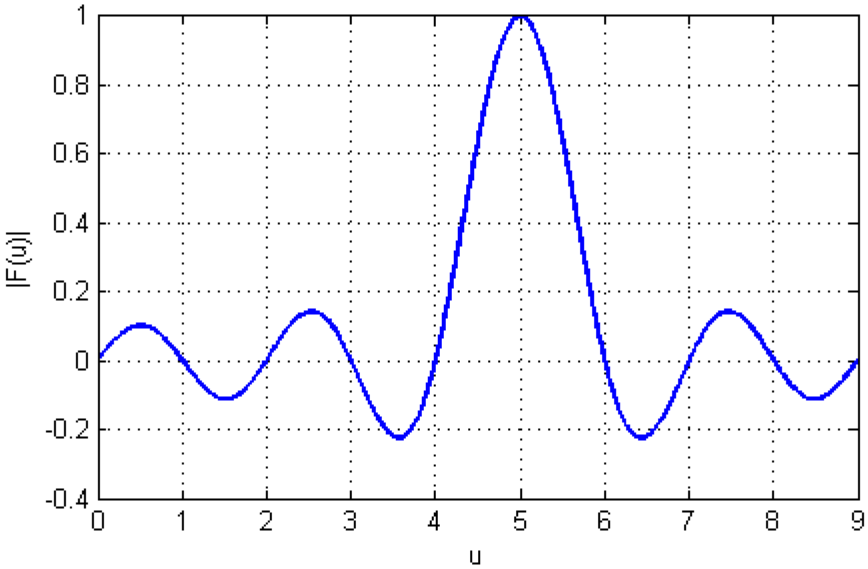
Taking the limit as $u \rightarrow \frac{M}{2}$ and using l'hopital's rule

$$|F(0)| = \lim_{u\rightarrow 0}\frac{cos(0)}{\cos(0)} = 1$$

The numerator will be zero at $\pi u - \frac{\pi M}{2} = n\pi \quad n = \pm 1, \pm 2, \pm 3, \ldots.$

$$u = n + M/2$$

For example for M = 10



Problem 4.

Given:

$$H(u,v) = Ae^{-\frac{u^2+v^2}{2\sigma^2}}$$

To calculate the inverse fourier transform:

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(u,v)e^{j2\pi(xu+yv)}du\,dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ae^{-\frac{u^2+v^2}{2\sigma^2}}e^{j2\pi(xu+yv)}du\,dv$$

$$= A\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}+j2\pi ux}du\right]}_{I_1}\underbrace{e^{-\frac{v^2}{2\sigma^2}+j2\pi vy}dv}_{I_2}$$

Let

$$I_1 = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}+j2\pi ux}du$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(u^2-j4\pi\sigma^2 ux)}du$$

Using the completing the square method (إكمال مربع)

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(u^2-j4\pi\sigma^2 ux-(2\pi)^2\sigma^4x^2+(2\pi)^2\sigma^4x^2)}du$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left((u+j2\pi\sigma^2 x)^2+(2\pi)^2\sigma^4x^2\right)}du$$

$$= e^{-\frac{(2\pi)^2\sigma^2x^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left((u+j2\pi\sigma^2 x)^2\right)}du$$

Using the substitution method (تكامل بالتعويض)

Let

$$r = u + j2\pi\sigma^2 x$$

$$dr = du$$

$$I_1 = e^{-\frac{(2\pi)^2\sigma^2x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr$$

Multiplying and dividing by $\sqrt{2\pi}\sigma$

$$I_1 = e^{-\frac{(2\pi)^2\sigma^2x^2}{2}} \sqrt{2\pi}\sigma \left[\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr\right]$$

$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr$ is the definition of the probability density function (Gaussian function), its integration from $-\infty$ to ∞ is equal to 1

$$I_1 = e^{-\frac{(2\pi)^2\sigma^2x^2}{2}} \sqrt{2\pi}\sigma$$

Using the same procedure for I_2

$$I_2 = e^{-\frac{(2\pi)^2\sigma^2y^2}{2}} \sqrt{2\pi}\sigma$$

Then

$$h(x,y) = AI_1I_2$$

$$h(x,y) = A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}$$

Problem 5.

Given:

$$h(x,y) = 2\pi A e^{-2\pi^2\sigma^2(x^2+y^2)}$$

To calculate the Fourier transform:

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) e^{-j2\pi(xu+yv)} dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi A e^{-2\pi^2\sigma^2(x^2+y^2)} e^{-j2\pi(xu+yv)} dx\, dy$$

$$= 2\pi A \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} e^{-2\pi\sigma^2x^2 - j2\pi ux} dx\right]}_{I_1} \underbrace{e^{-2\pi\sigma^2y^2 - j2\pi vy} dy}_{I_2}$$

Let

$$I_1 = \int_{-\infty}^{\infty} e^{-2\pi\sigma^2x^2 - j2\pi ux} dx$$

$$= \int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2\left(x^2+\frac{ju}{\pi\sigma^2}x\right)} dx$$

Using the completing the square method (إكمال مربع)

$$= \int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2\left(x^2+\frac{ju}{\pi\sigma^2}x-\frac{1}{4\pi^2\sigma^4}u^2+\frac{1}{4\pi^2\sigma^4}u^2\right)} dx$$

$$= e^{-\frac{1}{2\sigma^2}u^2} \int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2\left(x+\frac{j}{2\pi\sigma^2}u\right)^2} dx$$

Using the substitution method (تكامل بالتعويض)

Let

$$r = x + \frac{j}{2\pi\sigma^2} x$$

$$dr = du$$

$$I_1 = e^{-\frac{1}{2\sigma^2}u^2} \int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2 r^2} dx$$

To get the standard Gaussian form of the function, let $-2\pi^2\sigma^2 = -\frac{1}{2\Sigma^2}$ $\rightarrow$ $\Sigma = \frac{1}{2\pi\sigma}$

$$I_1 = e^{-\frac{1}{2\sigma^2}u^2} \int_{-\infty}^{\infty} e^{\frac{-r^2}{2\Sigma^2}} dx$$

Multiplying and dividing by $\sqrt{2\pi}\Sigma$

$$I_1 = e^{-\frac{1}{2\sigma^2}u^2} \sqrt{2\pi}\Sigma \left[\frac{1}{\sqrt{2\pi}\Sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\Sigma^2}} dr\right]$$

$\frac{1}{\sqrt{2\pi}\Sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\Sigma^2}} dr$ is the definition of the probability density function (Gaussian function), its integration from $-\infty$ to ∞ is equal to 1

$$I_1 = e^{-\frac{(2\pi)^2\sigma^2u^2}{2}} \sqrt{2\pi}\Sigma$$

Substituting for $\Sigma = \frac{1}{2\pi\sigma}$

$$I_1 = e^{-\frac{1}{2\sigma^2}u^2} \left(\frac{1}{\sqrt{2\pi}\sigma}\right)$$

Using the same procedure for I_2

$$I_2 = e^{-\frac{1}{2\sigma^2}v^2} \left(\frac{1}{\sqrt{2\pi}\sigma}\right)$$

Then

$$H(u,v) = 2\pi A I_1 I_2$$

$$h(x,y) = \frac{A}{\sigma^2} e^{\frac{(u^2+v^2)}{-2\sigma^2}}$$

Problem 6.

For the averaging filter for the $N4$ except the middle pixel, the result image is given by:

$$g(x,y) = \frac{1}{4}[f(x,y+1) + f(x+1,y) + f(x-1,y) + f(x,y-1)]$$

Using the property $f(x-x_o, y-y_o) = F(u,v)e^{-j2\pi\left(\frac{ux_o}{M}+\frac{vy_o}{N}\right)}$ to convert the previous equation to the frequency domain

$$G(u,v) = \frac{1}{4}\left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}}\right]F(u,v)$$

Since the relation between the result image and the original image in the frequency domain is

$$G(u,v) = H(u,v)F(u,v)$$

Where $H(u,v)$ is the applied filter then:

$$H(u,v) = \frac{1}{4}\left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}}\right]$$

$$H(u,v) = \frac{1}{2}\left[\cos\left(\frac{2\pi u}{M}\right) + \cos\left(\frac{2\pi v}{N}\right)\right]$$

Centering the filter in the frequency domain

$$H(u,v) = \frac{1}{2}\left[\cos\left(\frac{2\pi(u-\frac{M}{2})}{M}\right) + \cos\left(\frac{2\pi\left(v-\frac{N}{2}\right)}{N}\right)\right]$$

For a low pass filter the magnitude of the filter should be large at frequencies near the center (near u=M/2, v=N/2) and very small at frequencies near the edges (near u = 0,v=0 or u = M, v=N)

However it should be noted that since this filter only considers the 4-neighbourhood of the pixel, then it should only affect the frequencies at the vertical or horizontal direction

$$\left|H\left(\frac{M}{2},\frac{N}{2}\right)\right| = 1$$

$$\left|H\left(\frac{M}{2},0\right)\right| = \left|H\left(0,\frac{N}{2}\right)\right| = 0$$

$$|H(M,0)| = |H(0,N)| = 0$$

However if we consider $|H(0,0)| = |H(M,N)| = 1$…. i.e. the filter doesn't affect the frequencies at angle $45°$ which is expected.

Problem 7.

For the averaging filter for the $N8$ except the middle pixel, the result image is given by:

$$g(x,y) = \frac{1}{8}[f(x,y+1) + f(x+1,y) + f(x-1,y) + f(x,y-1) + f(x+1,y+1) + f(x+1,y-1) + f(x-1,y+1) + f(x-,y-1)]$$

Using the property $f(x-x_o, y-y_o) = F(u,v)e^{-j2\pi\left(\frac{ux_o}{M}+\frac{vy_o}{N}\right)}$ to convert the previous equation to the frequency domain

$$G(u,v) = \frac{1}{8}\left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}} + e^{-j2\pi\left(\frac{u}{M}+\frac{v}{N}\right)} + e^{-j2\pi\left(\frac{u}{M}-\frac{v}{N}\right)} + e^{-j2\pi\left(-\frac{u}{M}+\frac{v}{N}\right)} + e^{-j2\pi\left(-\frac{u}{M}-\frac{v}{N}\right)}\right] F(u,v)$$

Since the relation between the result image and the original image in the frequency domain is

$$G(u,v) = H(u,v)F(u,v)$$

Where $H(u,v)$ is the applied filter then:

$$H(u,v) = \frac{1}{8}\left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}} + e^{-j2\pi\left(\frac{u}{M}+\frac{v}{N}\right)} + e^{-j2\pi\left(\frac{u}{M}-\frac{v}{N}\right)} + e^{-j2\pi\left(-\frac{u}{M}+\frac{v}{N}\right)} + e^{-j2\pi\left(-\frac{u}{M}-\frac{v}{N}\right)}\right]$$

$$H(u,v) = \frac{1}{4}\left[\cos\left(\frac{2\pi u}{M}\right) + \cos\left(\frac{2\pi v}{N}\right) + \cos\left(2\pi\left(\frac{u}{M}+\frac{v}{N}\right)\right) + \cos\left(2\pi(\frac{u}{M}-\frac{v}{N}\right)\right]$$

Centering the filter in the frequency domain

$$H(u,v) = \frac{1}{4}\left[\cos\left(\frac{2\pi(u-\frac{M}{2})}{M}\right) + \cos\left(\frac{2\pi\left(v-\frac{N}{2}\right)}{N}\right) + \cos\left(2\pi\left(\frac{u-\frac{M}{2}}{M}+\frac{v-\frac{N}{2}}{N}\right)\right) + \cos\left(2\pi(\frac{u-\frac{M}{2}}{M}-\frac{v-\frac{N}{2}}{N}\right)\right]$$

For a low pass filter the magnitude of the filter should be large at frequencies near the center (near u=M/2, v=N/2) and very small at frequencies near the edges (near u = 0,v=0 or u = M, v=N)

$$\left|H\left(\frac{M}{2},\frac{N}{2}\right)\right| = 1$$

$$\left|H\left(\frac{M}{2},0\right)\right| = \left|H\left(0,\frac{N}{2}\right)\right| = 0.5$$

$$|H(M,0)| = |H(0,N)| = 0.5$$

$$|H(0,0)| = |H(M,N)| = 0$$

As expected.

Problem8.

For the Laplacian filter, the result image is given by:

$$g(x,y) = f(x,y+1) + f(x+1,y) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Using the property $f(x-x_o, y-y_o) = F(u,v)e^{-j2\pi\left(\frac{ux_o}{M}+\frac{vy_o}{N}\right)}$ to convert the previous equation to the frequency domain

$$G(u,v) = \left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}} - 4\right] F(u,v)$$

Since the relation between the result image and the original image in the frequency domain is

$$G(u,v) = H(u,v)F(u,v)$$

Where $H(u,v)$ is the applied filter then:

$$H(u,v) = \left[e^{-\frac{j2\pi v}{N}} + e^{-\frac{j2\pi u}{M}} + e^{\frac{j2\pi u}{M}} + e^{\frac{j2\pi v}{N}} - 4\right]$$

$$H(u,v) = 2\left[\cos\left(\frac{2\pi u}{M}\right) + \cos\left(\frac{2\pi v}{N}\right) - 2\right]$$

Centering the filter in the frequency domain

$$H(u,v) = 2\left[\cos\left(\frac{2\pi(u-\frac{M}{2})}{M}\right) + \cos\left(\frac{2\pi\left(v-\frac{N}{2}\right)}{N}\right) - 2\right]$$

For a high pass filter the magnitude of the filter should be small at frequencies near the center (near u=M/2, v=N/2) and large at frequencies near the edges (near u = 0,v=0 or u = M, v=N)

However it should be noted that since this filter only considers the 4-neighbourhood of the pixel, then it should only affect the frequencies at the vertical or horizontal direction

$$\left|H\left(\frac{M}{2},\frac{N}{2}\right)\right| = 0$$

$$\left|H\left(\frac{M}{2},0\right)\right| = \left|H\left(0,\frac{N}{2}\right)\right| = 4$$

$$|H(M,0)| = |H(0,N)| = 4$$

$$|H(0,0)| = |H(M,N)| = 8$$

As expected.

Problem 10.

$$F_1(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1} f_1(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$F_2(u,v) = F_1(u,v)^*$$

$$= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(x,y)\,e^{+j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

$$f_3(x,y) = L^{-1}\left[\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(x,y)\,e^{+j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right]$$

$$= L^{-1}\left[\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f_1(-x,-y)\,e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}\right]$$

$$f_3(x,y) = f_1(-x,-y)$$

Which means that the image should be mirrored about the origin (Fliped)