4.5 Shaft Misallignments and Flexible Couplings

Mechanical systems always involve two or more shafts to transfer motion. There is always a finite accuracy with which the two shafts can be alligned in the axial direction. Any shaft misallignments will result in loads on the bearings of the shafts and cause vibration, hence reduce the life of the machinery. In order to reduce the vibration and life reducing effects of shaft misallignement, flexible couplings are used between shafts (Fig.4.13).
There are two main categories of flexible shaft couplings,

1. couplings for large power transfer between shafts and motors,
2. couplings for precision motion transfer at low powers between shafts and motors.

High precision motion systems include motors with very low friction and yet very delicate bearings. Such motors are very sensitive to shaft misallignments. The bearing failure of the motor as a result of excessive shaft mis-allignment is a very common reliablity problem. Therefore, in most high performance servo motor applications, the motor shaft is coupled to the load via a flexible coupling. The flexible couplings provide the ability to make the system more tolerant to the shaft misallignments. However, it comes at the cost of reduced stiffness of the mechanical system. Therefore, designers must make sure the stiffness of the coupling does not interfere with the desired motion bandwidth (especially in variable speed and cyclic positioning applications).
Couplings are rated by the following main parameters,

1. maximum and rated torque capacity,
2. torsional stiffness,
3. maximim allowed axial misallignment,
4. rotary inertia and mass of the coupling,
5. input and output shaft diameters,
6. input/output shaft connection method (set screw, clamped with keyway),
7. design type (bellow or helical coupling).

4.6 Actuator Sizing

Every motion axes is powered by an actuator. The actuator may be a electric, hydraulic or pneumatic power based. In any case, the size of the actuator refers to its power capacity and must be large enough to be able to move the axis under the given inertial and load force/torque conditions. If the actuator is undersized, the axis will not be able to deliver the desired motion, i.e. can not deliver desired acceleration or speed levels or over heats due to over loading. If the actuator is over sized, it will cost more and the motion axis will have slower bandwidth since as the actuator size gets larger (larger power levels), the bandwidth gets slower as a general rule. Therefore, it is important to properly size the actuators for a motion axis appropriately with a reasonable margin of safety. Along with determining the proper actuator size for an application, the size of a gear mechanism may need to be determined unless the actuator is directly coupled to the load (Fig.4.15). The focus of this section is on sizing a rotary electric motor type actuator. Same concepts can be used for other types of actuators.

The question of actuator sizing is the question of determining the following requirements for an axis under worst operating conditions (i.e. largest expected inertia and resistive load),

1. maximum torque (also called peak torque) required, T_{max},
2. rated (continuous) torque required, T_r,
3. maximum speed required, $\dot{\theta}_{max}$,
4. positioning accuracy required, $(\Delta\theta)$,
5. gear mechanism parameters: gear ratio, its inertial and resistive load (force/torque), stiffness, backlash characteristics.

Once the torque requirements are determined, then the amplifier current and power supply requirements are directly determined from them.

In general, accuracy and maximum speed requirements of the load dictate the gear ratio. Below, we will assume that a gear mechanism with an approporiate gear ratio is selected and focus on determining the actuator size. For a given application, the load motion requirements specify the desired positioning accuracy and maximum speed. Let us call that be Δx and $\dot{x}_{max}$. The desired positioning accuracy and maximum speed at the actuator (i.e. rotary motor) is determined by,

$$\theta = N \cdot x \quad (4.148)$$

$$\Delta\theta = N \cdot \Delta x \quad (4.149)$$

$$\dot{\theta} = N \cdot \dot{x}_{max} \quad (4.150)$$

A gear ratio range is defined by the minimum accuracy and the maximum speed requirement, i.e. in order to provide the desired accuracy for a motor with a given positioning accuracy, the gear ratio must be

$$N \geq \frac{\Delta\theta}{\Delta x} \quad (4.151)$$

In order to provide the desired maximum speed for a given maximum speed capacity of the motor (not to exceed the maximum speed capability of the motor), the gear ratio must be smaller than or equal to

$$N \leq \frac{\dot{\theta}_{max}}{\dot{x}_{max}} \tag{4.152}$$

Hence, the acceptable gear ratio range is defined by the accuracy and maximum speed requirement,

$$\frac{\Delta\theta}{\Delta x} \leq N \leq \frac{\dot{\theta}_{max}}{\dot{x}_{max}} \tag{4.153}$$

Some of the most commonly used motion conversion mechanisms (also called the gear mechanism) are shown in Fig.4.16. Notice that the gear mechanism adds inertia and possible load torque to the motion axis in addition to performing the gear reduction role and coupling the actuator to the load. In precision positioning applications, the first requirement that must be satisfied is the accuracy.

The actuator needs to generate torque/force in order to move two different categories of inertia and load (Fig.4.15),

1. load inertia and force/torque,

2. inertia (and any resistive force) of the actuator itself. For instance, an electric motor has a rotor with finite inertia and that inertia is important on how fast the motor can accelerate and decellerate in high cycle rate automated machine applications. Similarly, a hydraulic cylinder has a piston and large rod which has non negligable mass.

The torque/force and motion relationship for each axis is determined by the Newton's Second Law. Let us consider it for a rotary actuator. The same relationships follow for translational actuators by replacing the rotary inertia with mass, torque with force, and angular acceleration with translational acceleration ($\{J_T,\ \ddot{\theta},\ T_T\}$ replace with $\{m_T,\ \ddot{x},\ F_T\}$),

$$J_T \cdot \ddot{\theta} = T_T \tag{4.154}$$

where J_T is the total inertia reflected on the motor axis, T_T is the total net torque acting on the motor axis, and $\ddot{\theta}$ is angular acceleration. The *reflected* inertia or torque means the equivalent inertia or torque seen at the motor shaft after the gear reduction affect is taken into account.

There are three issues to determine for the actuator sizing (Fig.4.15),

1. determine the net inertia (it may be function of the position of the motion conversion mechanism, Fig.4.16),

2. determine the net load torques (it may be function of the position of the motion conversion mechanism and speed),

3. specify the desired motion profile.

Let us discuss the first item, detemination of inertia. Total inertia is the inertia of the rotary actuator and the reflected inertia,

$$J_T = J_m + J_{l,eff} \tag{4.155}$$

where $J_{l,eff}$ includes all the load inertias reflected on the motor shaft. For instance, in the case of a ball-screw mechanism, this includes the inertia of the flexible coupling (J_c) between the motor shaft and ball-screw, ball-screw inertia (J_{bs}), and load mass inertia (due to W_l),

$$J_{l,eff} = J_c + J_{bs} + \frac{1}{(2\pi p)^2}(W_l/g) \tag{4.156}$$

Notice that the total inertia that the actuator has to move is sum of the load (including motion transmission mechanism) and the inertia of the moving part of the actuator itself.

The total torque is the difference between the torque generated by the motor (T_m) minus the resistive load torques on the axis (T_l),

$$T_T = T_m - T_{l,eff} \tag{4.157}$$

where T_l represent the sum of all external torques. If the load torque is in the direction of assisting the motion, it will be negative, and net result will be the addition of two torques. The T_l may include friction (T_f), gravity (T_g), and process related torque and forces (i.e. an assemly application may require the mechanism to provide a desired force pressure, T_a), nonlinear motion related forces/torques if any (i.e Corriolis forces and torques, T_{nl}),

$$\begin{aligned} T_l &= T_f + T_g + T_a + T_{nl} && (4.158) \\ T_{l,eff} &= \frac{1}{N} \cdot T_L && (4.159) \end{aligned}$$

Notice that the friction torque may have constant and speed dependent component to represent the Coulomb and viscous friction, $T_f(\dot{\theta})$.

For actuator sizing purposes, these torques should be considered for the worst possible case. However, care should be exercised that too much safety margin in the worst case assumptions can lead to unnecessarily large actuator sizing. Once the friction, gravitational loading, task related forces and other nonlinear force coupling effects in articulated mechanisms are estimated, the mechanism kinematics is used to determine the reflected forces on the actuator axis. This reflection is a constant ratio for simple motion conversion mechanisms such as gear reducers, belt-pulley, lead-screw. For more complicated mechanisms such as linkages, cams and multi degrees of freedom mechanisms, the kinematic reflection relations for the inertias and forces are not constant. Again, these relationships can be handled using worst case assumptions in simpler forms, or using more detailed nonlinear kinematic model of the mechanism (see Section 4.7).

Finally, we need to know the desired motion profile of the axis as a function of time. Generally, we assume a worst case cyclic motion. The most common motion profile used is a trapezoidal velocity profile as a function of time (Fig.4.14). The typical motion include a constant acceleration period, then a constant speed period, then a constant decelleration period, and a dwell (zero speed) period.

$$\dot{\theta} = \dot{\theta}(t) \quad ; \quad 0 \leq t \leq t_{cyc} \tag{4.160}$$

Once the inertias, load torques, and desired motion profile are known, the required torque as a function of time during a cycle of the motion can be determined from

$$\begin{aligned} T_m(t) &= J_T \cdot \ddot{\theta}(t) + T_l && (4.161) \\ &= J_T(\theta) \cdot \ddot{\theta}(t) + T_f(\dot{\theta}) + T_g(\theta) + T_a(t) + T_{nl}(\theta, \dot{\theta}) && (4.162) \end{aligned}$$

Notice that before we calculate the torque requirements, we need to guess the inertia of the actuator itself which is not known yet. Therefore, this calculation may need to be iterated few times.

Once the required torque profile is known as a function of time, two sizing values are determined from it: the maximum and root-mean square (RMS) value of the torque,

$$\begin{aligned} T_{max} &= max(T_m(t)); && (4.163) \\ T_r &= T_{rms} = (\frac{1}{t_{cyc}} \int_0^{t_{cyc}} T_m(t)^2 \; dt)^{1/2} && (4.164) \end{aligned}$$

From the desired motion profile specification, we determine the maximum speed the actuator must deliver using the kinematic relations. In order to design an optimal motion control axis, the actuator sizing and the motion conversion mechanism (effective gear ratio) should be considered together. It may be possible that a very small gear ratio may require a motor with very large torque requirement, and yet run at very low speeds, hence make use of a small part of the power capacity of the motor. An increased gear ratio would then require smaller torque motor and that the motor would operate at higher speeds on the average, hence making use of the available power of the motor more.

Once the torque requirements are known, the drive current and power supply voltage requirements can be directly determined for a given electric motor-drive system. Similarly, for hydraulic actuators, once the force requirements are determined, we would pick the supply pressure, and determine the diameter of the linear cylinder. The speed requirement would determine the flow rate. Once these are known, then the size of the valve and pump can be determined.

Actuator Sizing Algorithm (Fig.4.15, 4.16):

1. Define the geometric relationship between the actuator and load. In other words, select the type of motion transmission mechanism between the motor and the load (N).

2. Define the inertia and torque/force characteristics of the load and the transmission mechanisms, i.e. define the inertia of the tool as well as the inertia of the gear reducer mechanisms (J_l, T_l).

3. Define the desired cyclic motion profile in the form of load speed versus time ($\dot{\theta}_l(t)$).

4. Using the reflection equations developed above, calculate the reflected load inertia and torque/forces (J_{eff}, T_{eff}) that will effectively act on the actuator shaft as well as the desired motion at the actuator shaft ($\dot{\theta}_m(t) = \dot{\theta}_{in}(t)$).

5. Guess an actuator/motor inertia from an available list (or make the first calculation with zero motor inertia assumption), and calculate the torque history, $T_m(t)$, for the desired motion cycle. Then calculate the peak torque and RMS torque from $T_m(t)$.

6. Check if the actuator meets the required performance in terms of peak and RMS torque, and maximum speed capacity ($T_p, T_{rms}, \dot{\theta}_{max}$). If the above selected actuator/motor from the available list does not meet the requirements (i.e. too small or too large), repeat the previous step by selecting a different motor.

7. Continuous torque rating of most servo motors is given for $25^0 C$ ambient temperature and an aluminum face mount for heat dissipation considerations. If the nominal ambient temperature is different than $25^0 C$, the continuous (RMS) torque capacity of the motor should be derated using the following equation for a temperature,

$$T_{rms} = T_{rms}(25^0 C)\sqrt{(155 - Temp^o C)/130} \tag{4.165}$$

   If the T_{rms} rating is exceeded, the temperature of the motor winding will increase proportionally. If the temperature rise is above the rated temperature for the winding insulation of the motor, the motor will be damaged permanently.

4.6.1 Inertia Match Between Motor and Load

The ratio of the motor's rotor inertia and the reflected load inertia is always a concern in high performance motion control applications. It is a rule of thumb that the ratio of motor inertia to load inertia should be between one-to-one and upto one-to-ten,

$$\frac{J_m}{J_l/N^2} = \frac{1}{1} \quad \sim \quad \frac{1}{10} \tag{4.166}$$

The one-to-one match is considered the optimal match. Below, we show that one-to-one inertial match is *optimal* only in ideal case where the motor drives a purely inertial load and that this inertia ratio results in *minimum heating of the motor*.

Let us consider the case that the motor is coupled to a purely inertial load through an effective gear ratio. The torque and motion relationship is

$$T_m(t) = (J_m + \frac{1}{N^2} J_l) \cdot \ddot{\theta}_m \tag{4.167}$$

$$= (J_m + \frac{1}{N^2} J_l) \cdot N \cdot \ddot{\theta}_l \tag{4.168}$$

The minimal heating occurs when the required torque is minimized, since torque is proportional to current and heat generation is related to current ($P_{elec} = R \cdot i^2$, where P_{elec} is the electric power dissipated at the motor windings due to its electrical resistance R and current i). The minimum torque occurs at the gear ratio where the derivative of T_m with respect to N is equal to zero,

$$\frac{d}{dN}(T_m) = (J_m + \frac{1}{N^2} J_l)\ddot{\theta}_l + (\frac{-2N}{N^4} J_l) \cdot N\ddot{\theta}_l \tag{4.169}$$

$$= \ddot{\theta}_l \cdot (J_m - \frac{1}{N^2} J_l) \tag{4.170}$$

$$= 0 \tag{4.171}$$

Therefore, the optimal gear ratio between the motor and a purely inertial load which minimizes the torque requirements (hence, the heating), is

$$J_m = \frac{1}{N^2} J_l \tag{4.172}$$

$$= J_{l,reflected} \tag{4.173}$$

It is important to note that this ideal inertia match (1:1) between motor's rotor inertia and reflected load inertia is optimal only for purely inertial loads. In applications where the load may be dominated by friction or other application related load torque or forces, the ideal inertia match may not be a good design.

Example: Consider a rotary motion axis driven by an electric servo motor. The rotary load is directly connected to the motor shaft without any gear reducer. The rotary load is a solid cylindrical shape made of steel material, $d = 3.0\ in,\ l = 2.0\ in, \rho = 0.286\ lb/in^3$. The desired motion of the load is a periodic motion (Fig.4.15). The total distance to be traveled is 1/4 of a revolution. The period of motion is $t_{cyc} = 250\ msec$, and dwell portion of it is $t_{dw} = 100\ msec.$, and the remaining part of the cycle time is equally divided between acceleration, constant speed and deceleration periods, $t_a = t_r = t_d = 50\ msec$. Determine the required motor size for this application.

This example matches the ideal model shown in Fig.4.15. The load inertia is

$$J_l = \frac{1}{2} \cdot m \cdot (d/2)^2 \quad (4.174)$$
$$= \frac{1}{2} \cdot \rho \cdot \pi \cdot (d/2)^2 \cdot l \cdot (d/2)^2 \quad (4.175)$$
$$= \frac{1}{2} \cdot \rho \cdot \pi \cdot l \cdot (d/2)^4 \quad (4.176)$$
$$= \frac{1}{2} \cdot (0.286/386) \cdot \pi \cdot 2.0 \cdot (3/2)^4 \ [lb\ in\ sec^2] \quad (4.177)$$
$$= 0.0118 \ [lb\ in\ sec^2] \quad (4.178)$$

Let us assume that we will pick a motor which has an rotor inertia same as the load so that there is an ideal load and motor inertia match, $J_m = 0.0118 \ [lb \cdot in \cdot sec^2]$. The acceleration, top speed and deceleration rates are calculated from the kinematic relationships,

$$\theta_a = \frac{1}{2}\dot{\theta} \cdot t_a = \frac{1}{4} \cdot \frac{\pi}{2} \quad (4.179)$$
$$\dot{\theta} = 2 \cdot \theta_a / t_a = 2 \cdot (1/4) \cdot (\pi/2) \ [rad]/(0.05 \ [sec]) \quad (4.180)$$
$$= 80\pi/16 \ [rad/sec] = 40/16[rev/sec] = 2400/16[rev/min] = 150[rev/min] \quad (4.181)$$
$$\ddot{\theta}_a = \dot{\theta}_a / t_a = (80\pi/16)(1/0.05) = 1600\pi/16[rad/sec^2] = 100\ \pi \ [rad/sec^2] \quad (4.182)$$
$$\ddot{\theta}_r = 0.0 \quad (4.183)$$
$$\ddot{\theta}_d = -100\ \pi \ [rad/sec^2] \quad (4.184)$$
$$\ddot{\theta}_{dw} = 0.0 \quad (4.185)$$

The required torque to move the load through the desired cyclic motion can be calculated as follows,

$$T_a = (J_m + J_l) \cdot \ddot{\theta} = (0.0118 + 0.0118) \cdot (100\pi) = 7.414 \ [lb\ in] \quad (4.186)$$
$$T_r = 0.0 \quad (4.187)$$
$$T_d = (J_m + J_l) \cdot \ddot{\theta} = (0.0118 + 0.0118) \cdot (-100\pi) = -7.414 \ [lb\ in] \quad (4.188)$$
$$T_{dw} = 0.0 \quad (4.189)$$

Hence, the peak torque requirement is

$$T_{max} = 7.414[lb\ in] \quad (4.190)$$

and the RMS torque requirement is

$$T_{rms} = (\frac{1}{0.250}(T_a^2 \cdot t_a + T_r^2 \cdot t_r + T_d^2 \cdot t_d + T_{dw}^2 \cdot t_{dw}))^{1/2} \quad (4.191)$$

$$= (\frac{1}{0.250}(7.414^2 \cdot 0.05 + 0.0 \cdot 0.05 + (-7.414)^2 \cdot 0.05 + 0.0 \cdot 0.1))^{1/2} \quad (4.192)$$

$$= 4.689 \ [lb \ in] \quad (4.193)$$

Therefore, a motor which has rotor inertia of about $0.0118 [lb \ in \ sec^2]$, maximum speed capabaility of 150 $[rev/min]$ or better, peak and RMS torque rating in the range of 8.0 $[lb \ in]$ and 5.0 $[lb \ in]$ range would be sufficient for the task.

This design may be improved further by the following consideration. The top speed of the motor is only 150 *rpm*. Most electric servo motors runs in the 1500 *rpm* to 5000 *rpm* range where they deliver most of their power capacity, while maintaining a constant torque capacity upto these speeds. As a result, it is reasonable to consider a gear reducer between the motor shaft and the load in the range of 10 : 1 to 20 : 1 and repeat the sizing calculations. This will result in a motor that will run at a higher speed and will have lower torque requirements.

4.7 Homogeneous Transformation Matrices

The geometric relationships in simple one degree of freedom mechanisms can be derived using basic vector algebra. The derivation of geometric relations for multi degrees of freedom mechanisms, such as the robotic mechanisms, is rather difficult using three dimensional vector algebra. The so called (4x4) homogeneous transformation matrices are very powerful matrix methods to describe the geometric relations. They are used to describe the geometric relations of a mechanism between the absolute values of

1. displacement variables,
2. the relations between the incremental changes in displacements, and
3. force and torque transmission through the mechanism.

The position and orientation of a three dimensional object with respect to a reference frame can be uniquely described by the position coordinates of a point on it (three components of position information in three dimensional space) and the orientation of it (described by three angles). The position coordinates are associated with a point and are unique for a given point with respect to a reference coordinate frame. Orientation is associated with an object, not a point. The best way to describe the position and orientation of an object is to attach a coordinate frame to the object, and describe the orientation and the origin coordinates of the attached coordinate frame with respect to a reference frame. For instance, the position and orientation of a tool held by the gripper of a robotic manipulator can be described by a coordinate frame attached to the tool (Fig.4.17). The position coordinates of the origin of the attached coordinate frame and its orientation with respect to another reference frame also describes the position and orientation of the tool on which the frame is attached to.

The transformation of an object between any two different orientations can be accomplished by a sequence of three independent rotations. However, the number and sequence of rotation angles to go from one orientation to another is not unique. There are many possible rotation combinations to make a desired orientation change. For instance, an orientation change between any two different orientation of two coordinate frames can be accomplished by a sequence of three angles such as

1. roll, picth and yaw angles,
2. Z,Y,X Euler angles,
3. X,Y,Z Euler angles.

There are 24 different possible combinations of a seqeunce of three angles to go from one orientation to another. *Finite rotations are not commutative. Infinitesimal rotations are commutative.* That means, the order of a seqeuence of finite rotations makes a difference in the final orientation. For instance, making a 90^0 rotation about X axis followed by another 90^0 rotation about Y axis results in a different orientation than that of making a 90^0 rotation about Y axis followed by another 90^0 rotation about X axis. However, if the rotations are infinitesimal, the order does not matter.

The 4x4 homogeneous transformation matrices describe the position of a point on an object and the orientation of the object in three dimensional space using a 4x4 matrix. The first 3x3 portion of the matrix is used to define the orientation of a coordinate frame fixed to the object with respect to another reference coordinate frame. The last colum of the matrix is used to describe the position of the origin of the coordinate frame fixed to the object with respect to the origin of the reference coordinate frame. The last row of the matrix is [0 0 0 1]. A general 4x4 homogeneous transformation matrix T has the following form (Fig.4.18),

$$T = \begin{bmatrix} e_{11} & e_{12} & e_{13} & x_A \\ e_{21} & e_{22} & e_{23} & y_A \\ e_{31} & e_{32} & e_{33} & z_A \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.194)$$

It describes the position and orientation of the coordinate frame A with respect to the coordinate frame 0. The columns of the (3x3) portion of the matrix which contain the orientation information are the cosine angles between the unit vectors of the coordinate frames. Notice that, even though we know that the orientation of one coordinate frame with respect to another can be described by three angles, the general form of the rotation portion of the (4x4) transformation matrix (the (3x3) portion) requires nine parameters. However, they are not all independent. There are six constraints between them, leaving three independent variables (Fig.4.18). The six constraints are

$$\vec{e}_1 \cdot \vec{e}_1 = 1.0 \quad (4.195)$$

$$\vec{e}_2 \cdot \vec{e}_2 = 1.0 \quad (4.196)$$

$$\vec{e}_3 \cdot \vec{e}_3 = 1.0 \tag{4.197}$$
$$\vec{e}_1 \cdot \vec{e}_2 = 0 \tag{4.198}$$
$$\vec{e}_1 \cdot \vec{e}_3 = 0 \tag{4.199}$$
$$\vec{e}_2 \cdot \vec{e}_3 = 0 \tag{4.200}$$

where

$$\vec{e}_i = e_{1i}\vec{i} + e_{2i}\vec{j} + e_{3i}\vec{k} \quad ; \quad i = 1, 2, 3 \tag{4.201}$$

are the unit vectors along each of the axes of the attached coordinate frame expressed in terms of its components in the unit vectors of the other coordinate frame. The use of cosine angles in describing the orientation of one coordinate frame with respect to another one is very convenient way to determine the elements of the matrix.

The 4x4 homogeneous transformation matrices are the most widely accepted and powerful (if not computationally most efficient) method to describe kinematic relations. The algebra of transformation matrices follows the basic matrix algebra. Let us consider three coordinate frames numbered 1, 2, 3, 4 and a point A on the object where the position coordinates of the point with respect to the third coordinate frame is described by r_{4A} (Fig.**??**). The description of the coordinate frame 4 (position of its origin and orientation) can be expressed as,

$$T_{04} = T_{01} \cdot T_{12} \cdot T_{23} \cdot T_{34} \tag{4.202}$$

where T_{02}, T_{12}, T_{23} and T_{34} are the description of the origin position coordinates and orientations of the axes of coordinate frames 1 with respec to 0, 2 with respect to 1 and that of 3 with respect to 2. The position coordinate vector of point A can be expressed with respect to coordinates 2 and 3 as follows (Fig.4.18),

$$r_{3A} = T_{34} \cdot r_{4A} \tag{4.203}$$
$$r_{2A} = T_{23} \cdot r_{3A} = T_{23} \cdot T_{34} \cdot r_{4A} \tag{4.204}$$
$$r_{1A} = T_{12} \cdot r_{2A} = T_{12} \cdot T_{23} \cdot T_{34} \cdot r_{4A} \tag{4.205}$$
$$r_{0A} = T_{01} \cdot r_{1A} = T_{01} \cdot T_{12} \cdot T_{23} \cdot T_{34} \cdot r_{4A} \tag{4.206}$$

where, $r_{4A} = [\ x_{4A} \quad y_{4A} \quad z_{4A} \quad 1\ ]^T$. The $r_{3A}, r_{2A},\ r_{1A},\ r_{0A}$ are similarly defined. Notice that T_{12} is the description (position coordinates of the origin and the orientation of the axes of coordinate system 2 with respect to coordinate system 1) of coordinate system 2 with respect to coordinate system 1. Then the reverse description, that is the description of coordinate system 1 with respect to coordinate system 2 is the inverse of the previous transformation matrix,

$$T_{21} = T_{12}^{-1} \tag{4.207}$$

The (4x4) transformation matrix has a special form and the inversion of the matrix also has a special result. Let

$$T_{12} = \begin{bmatrix} & R_{12} & & p_A \\ - & - & - - - - - & \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.208}$$

Then, the inverse of this matrix can be shown as

$$T_{12}^{-1} = \begin{bmatrix} & R_{12}^T & & -R_{12}^T \cdot p_A \\ - & - & - - - - - & \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.209}$$

Also notice that the order of transformations is important (multiplication of matrices are dependent on the order),

$$T_{12} \cdot T_{23} \neq T_{23} \cdot T_{12} \tag{4.210}$$

For a general purpose multi degrees of freedom mechanism, such as a robotic manipulator, the relationship between a coordinate frame attached to the tool (the coordinates of its origin and orientation) with respect to a fixed reference frame at the base can be expressed as a seqeunce of transformation matrices where each transformation matrix is a function of the one of the axis position variables. For instance, for a four degrees of freedom robotic manipulator, the coordinate system at the wrist joint can be described with respect to base as (Fig.**??**)

$$T_{04} = T_{01}(\theta_1) \cdot T_{12}(\theta_2) \cdot T_{23}(\theta_3) \cdot T_{34}(\theta_4) \tag{4.211}$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ are the position of axes driven by motors. Any given position vector relative to the forth coordinate frame (r_{4A}) can be expressed with respect to base as follows,

$$r_{0A} = T_{04} \cdot r_{4A} \tag{4.212}$$

$$= T_{01}(\theta_1) \cdot T_{12}(\theta_2) \cdot T_{23}(\theta_3) \cdot T_{34}(\theta_4) \cdot r_{4A} \tag{4.213}$$

where $r_{4A} = [\ x_{4A},\ y_{4A},\ z_{4A},\ 1\ ]^T$ the three coordinates of the point A with respect to the coordinate frame 4.

Denavit-Hartenberg method [Denavit 67] defines a standard way of attaching coordinate frames to a robotic manipulator such that only four numbers (one variable, three constant parameters) are needed per one-degree of freedom joint to represent the kinematic relationships.

In generic terms, the relationship between the coordinates of the tool and the joint displacement variables can be expressed as,

$$\underline{x} = \underline{f}(\underline{\theta}) \tag{4.214}$$

The vector variable $\underline{x}$ represent the cartesian coordinates of the tool (i.e. position coordinates x_P, y_P, z_P in a given coordinate frame and oriention angles where three angles can be used to describe the orientation). The description of the position coordinates of a point with respect to a given reference frame is unique. However, the orientation of an object with respect to a reference coordinate frame can be desribed by many different possible combinations of angles. Hence, the orientation description is not unique. The vector variable $\underline{\theta}$ represent the joint variables of the robotic manipulator, i.e. for a six joint robot $\underline{\theta} = [\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6\ ]^T$.

The $\underline{f}(\underline{\theta})$ is called the *forward kinematics* of the mechanism, which is a vector nonlinear function of joint variables,

$$\underline{f}(\underline{\theta}) = [f_1(\underline{\theta}), f_2(\underline{\theta}), f_3(\underline{\theta}), f_4(\underline{\theta}), f_5(\underline{\theta}), f_6(\underline{\theta})]^T \tag{4.215}$$

The inverse relationship, that is the geometric function which defines the axis positions as function of tool position and orientation is called the *inverse kinematics* of the mechanism,

$$\underline{\theta} = \underline{f}^{-1}(\underline{x}) \tag{4.216}$$

The inverse kinematics function may not be possible to express in one analytical closed form for every mechanism. It must be determined for each special mechanisms on case by case basis. For a six revolute joint manipulator, a sufficient condition for the existence of the inverse kinematic solution in analytical form is that three consecutive joint axes must intersect at a point. Forward and inverse kinematic functions of a mechanism relate the joint positions to the tool positions.

The *differential* relationships between joint axis variables and tool variables are obtained by taking the differential of the forward kinematic function. The resultant matrix that relates the differential values of joint and tool position variables (in other words, it relates the velocities of joint axes and tool motion) is called the *Jacobian matrix* of the mechanism (Fig.4.14).

$$\dot{\underline{x}} = \frac{d\underline{f}(\underline{\theta})}{d\underline{\theta}} \cdot \dot{\underline{\theta}} \tag{4.217}$$

$$= J\dot{\underline{\theta}} \tag{4.218}$$

where the J matrix is called the Jacobian of the mechanism. Each element of the Jacobian matrix is defined as

$$J_{ij} = \frac{\partial f_i(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)}{\partial \theta_j} \tag{4.219}$$

where i and $j = 1, 2, .., n$, where n is the number of joint variables. For a six degrees of freedom mechanism, $n = m = 6$. If the mechanism has less than six degrees of freedom, the Jacobian matrix is not a square matrix. Likewise, the inverse of the jacobian, J^{-1} relates the changes in the tool position to the changes in the axis displacements,

$$\underline{\dot{\theta}} = J^{-1} \cdot \underline{\dot{x}} \tag{4.220}$$

If the jacobian is not invertible in certain positions, these positions are called the geometric singularities of the mechanism. It means that at these locations, there are some directions that the tool can not move no matter what the change is in the joint variables. In other words, no joint axis variable combination can generate a motion in certain directions at a singularity point. A robotic manipulator may have many singularity points in its workspace. The geometric singularity is directly function of the mechanical configuration of the manipulator. There are two groups of singularities,

1. workspace boundary: a given manipulator has a finite span in three dimensional space. The locations that the manipulator can reach is called the *workspace*. At the boundary of workspace, the manipulator tip can not move out, because it reached its limits of reach. Hence, all points in workspace boundary are singularity points since at these points there are directions along which manipulator tip can not move.

2. workspace interior points: these singularity points are inside the workspace of the manipulator. Such singularity points depend on the manipulator geometry and generally occur when two or more joints line up.

The same jacobian matrix also describes the relationship between the torques/forces at the controlled axes and the force/torque exprienced at the tool. Let the tool force be $\underline{Force}$ and the corresponding tool position differential displacement be $\delta\underline{x}$. The differential work done is

$$Work = \underline{\delta x}^T \cdot \underline{Force} \tag{4.221}$$

Note that the jacobian relationship

$$\underline{\delta x} = J \cdot \underline{\delta \theta} \tag{4.222}$$

and the equivalent work done by the corresponding torques at the controlled axes can be expressed as

$$\begin{aligned} Work &= \underline{\delta\theta}^T \cdot \underline{Torque} & (4.223)\\ &= \delta x^T \cdot Force & (4.224)\\ &= (J \cdot \delta\theta)^T \cdot Force & (4.225) \end{aligned}$$

Hence, the force-torque relationship between the tool and joint variables is

$$\underline{Torque} = J^T \cdot \underline{Force} \quad (4.226)$$

and the inverse relationship is

$$\underline{Force} = (J^T)^{-1} \cdot \underline{Torque} \quad (4.227)$$

Notice that, at the singular configurations of the mechanism, those configuration of the mechanism at which the inverse of the Jacobian matrix does not exist, there are some force directions at the tool which does not result in any change in the axes torques. They only result in reaction forces in the linkage structure, but not in the actuation axes. Another interpretation of this result is that there are some directions of tool motion that we can not generate force no matter what combination of torques are applied at the joints.

There are different methods for the calculation of the jacobian matrix for a mechanism [Paul81, Craig89, Orin84]. The inverse jacobian matrix can be either obtained analytically in symbolic form or calculated numerically off-line or on-line (in real-time). However, real-time numerical inverse calculations present a problem both in terms of the computational load and the possible numerical stability problems around the singularities of the mechanism. The decision regarding the jacobian matrix and its inverse computations in real-time should be made on mechanism by mechanism basis.

Example

Consider two coordinate frames numbered 1 and 2 as shown in Fig.4.19. Let the origin coordinates of second coordinate frame have the following coordinates, $r_{1A} = [\ x_{1A}\ \ y_{1A}\ \ z_{1A}\ \ 1]^T = [\ -0.5\ \ 0.5\ \ 0.0\ \ 1]^T$. Orientations of axes are such that X_2 is parallel to Y_1, Y_2 is parallel but in opposite direction to X_1, and Z_2 has the same direction as Z_1. Determine the vector description of point P whose coordinates are given in second frame as $r_{2P} = [\ x_{2P}\ \ y_{2P}\ \ z_{2P}\ \ 1]^T = [\ 1.0\ \ 1.0\ \ 0.0\ \ 1]^T$.

Using the homogeneous transformation matrix relationship between coordinate frames 1 and 2,

$$r_{1P} = T_{12} \cdot r_{2P} \tag{4.228}$$

where the T_{12} is described by the orientation of the second coordinate frame and position coordinates of its origin with respect to first coordinate frame.

$$T_{12} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & x_{1A} \\ e_{21} & e_{22} & e_{23} & y_{1A} \\ e_{31} & e_{32} & e_{33} & z_{1A} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.229}$$

$$= \begin{bmatrix} 0.0 & -1.0 & 0.0 & -0.5 \\ 1.0 & 0.0 & 0.0 & 0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.230}$$

Notice that the orientation portion of the matrix is the coefficients of the relationships between the unit vectors,

$$\vec{i} = Cos\theta_{11} \cdot \vec{e}_1 + Cos\theta_{12} \cdot \vec{e}_2 + Cos\theta_{13} \cdot \vec{e}_3 \tag{4.231}$$
$$= e_{11} \cdot \vec{e}_1 + e_{12} \cdot \vec{e}_2 + e_{13} \cdot \vec{e}_3 \tag{4.232}$$
$$= 0.0 \cdot \vec{e}_1 + (-1.0) \cdot \vec{e}_2 + 0.0 \cdot \vec{e}_3 \tag{4.233}$$
$$\vec{j} = Cos\theta_{21} \cdot \vec{e}_1 + Cos\theta_{22} \cdot \vec{e}_2 + Cos\theta_{23} \cdot \vec{e}_3 \tag{4.234}$$
$$= e_{21} \cdot \vec{e}_1 + e_{22} \cdot \vec{e}_2 + e_{23} \cdot \vec{e}_3 \tag{4.235}$$
$$= 1.0 \cdot \vec{e}_1 + (0.0) \cdot \vec{e}_2 + (0.0) \cdot \vec{e}_3 \tag{4.236}$$
$$\vec{k} = Cos\theta_{31} \cdot \vec{e}_1 + Cos\theta_{32} \cdot \vec{e}_2 + Cos\theta_{33} \cdot \vec{e}_3 \tag{4.237}$$
$$= e_{31} \cdot \vec{e}_1 + e_{32} \cdot \vec{e}_2 + e_{33} \cdot \vec{e}_3 \tag{4.238}$$
$$= (0.0) \cdot \vec{e}_1 + (0.0) \cdot \vec{e}_2 + (1.0) \cdot \vec{e}_3 \tag{4.239}$$

Hence,

$$r_{1P} = T_{12} \cdot r_{2P} \tag{4.240}$$
$$= [\ -1.5 \ \ 1.5 \ \ 0.0 \ \ 1\ ]^T \tag{4.241}$$

Example

The purpose of this example is to illustrate graphically that the order of a sequence of finite rotations is important. If we change the order of rotations, the final orientation is different (Fig.4.20). In other words, $T_1 \cdot T_2 \neq T_2 \cdot T_1$. Let T_1 represent a rotation about X axis by 90^o, and T_2 represent a rotation about the Y axis by 90^o. Figure 4.20 shows the sequence of both T_1 followed by T_2 and T_2 followed by T_1. The resulting final orientations are different since the order of rotations are different. Finite rotations are not commutative. We can show this algebraically for this case.

$$T_1 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.242}$$

$$T_2 = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.243}$$

Clearly, $T_1 \cdot T_2 \neq T_2 \cdot T_1$.

Example

Consider the geometry of a two-link robotic manipulator (Fig.4.21). The geometric parameters (link lengths l_1 and l_2) and joint variables are shown in the figure.

1. Derive the tip position P coordinates as function of joint variables (forward kinematic relations).
2. Derive the Jacobian matrix that relates the joint velocities to tip position coordinate velocities.
3. If there is a load with weight W at the tip, determine the necessary torques at the joints 1 and 2 in order to balance the load.

In this example, we are asked to determine the following relations,

$$\underline{x} = \underline{f}(\underline{\theta}) \tag{4.244}$$

$$\underline{\dot{x}} = J(\theta)(\underline{\dot{\theta}}) \tag{4.245}$$

$$\underline{Torque} = J^T(\theta)(\underline{Force}) \tag{4.246}$$

Let us attach three coordinate frames to the manipulator. Coordinate frame 0 is attached to the base and fixed, coordinate frame 1 is attached at joint 1 to link 1 (moves with link 1), coordinate frame 2 is attached at joint 2 to link 2 (moves with link 2). The position vector of the tip with respect to coordinate frame 2 is simple, $r_{2P} = [\, l_2 \;\; 0 \;\; 0 \;\; 1]^T$. The transformation matrices between the three coordinate frames are functions of θ_1 and θ_2 as follows,

$$T_{01} = \begin{bmatrix} Cos(\theta_1) & -Sin(\theta_1) & 0.0 & 0.0 \\ Sin(\theta_1) & Cos(\theta_1) & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.247}$$

$$T_{12} = \begin{bmatrix} Cos(\theta_2) & -Sin(\theta_2) & 0.0 & l_1 \\ Sin(\theta_2) & Cos(\theta_2) & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.248}$$

Hence, the tip vector description in base coordinates,

$$r_{0P} = [\, x_{0P} \;\; y_{0P} \;\; 0 \;\; 1]^T = T_{01} \cdot T_{12} \cdot r_{2P} \tag{4.249}$$

The vector components of r_{0P} can be expressed as individual functions for more clarity,

$$x_{0P} = x_{0P}(\theta_1, \theta_2; l_1, l_2) = l_1 \cdot Cos(\theta_1) + l_2 \cdot Cos(\theta_1 + \theta_2) \tag{4.250}$$

$$y_{0P} = y_{0P}(\theta_1, \theta_2; l_1, l_2) = l_1 \cdot Sin(\theta_1) + l_2 \cdot Sin(\theta_1 + \theta_2) \tag{4.251}$$

$$z_{0P} = 0.0 \tag{4.252}$$

The Jacobian matrix for this case can simply be determined by taking the derivative of the forward kinematic relations with respect to time and express the equation in the matrix form to find the Jacobian matrix.

$$\dot{x}_{0P} = \frac{d}{dt}(x_{0P}(\theta_1, \theta_2; l_1, l_2)) = J_{11}\dot{\theta}_1 + J_{12}\dot{\theta}_2 \tag{4.253}$$

$$\dot{y}_{0P} = \frac{d}{dt}(y_{0P}(\theta_1, \theta_2; l_1, l_2)) = J_{21}\dot{\theta}_1 + J_{22}\dot{\theta}_2 \tag{4.254}$$

where it is easy to show that the elements of the Jacobian matrix J

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{4.255}$$

$$= \begin{bmatrix} -l_1 \cdot Sin(\theta_1) - l_2 \cdot Sin(\theta_1 + \theta_2) & -l_2 \cdot Sin(\theta_1 + \theta_2) \\ l_1 \cdot Cos(\theta_1) + l_2 \cdot Cos(\theta_1 + \theta_2) & l_2 \cdot Cos(\theta_1 + \theta_2) \end{bmatrix} \tag{4.256}$$

The torque needed to balance a weight load, $F = [F_x \quad F_y]^T = [0 \quad -W]^T$, at the tip is determined by

$$\begin{bmatrix} Torque_1 \\ Torque_2 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \end{bmatrix} \begin{bmatrix} 0.0 \\ -W \end{bmatrix} \tag{4.257}$$

which shows the necessary static torque at each joint to balance a weight at the tip for different positions of the manipulator. Notice that since the Jacobian matrix is 2x2, it is relatively simple to obtain the inverse Jacobian in analytical form.

$$J^{-1} = \frac{1}{l_1 \cdot l_2 \cdot Sin(\theta_2)} \begin{bmatrix} l_2\ Cos(\theta_1 + \theta_2) & l_2 \cdot Sin(\theta_1 + \theta_2) \\ -l_1\ Cos(\theta_1) - l_2\ Cos(\theta_1 + \theta_2) & l_1\ Sin(\theta_1) - l_2 \cdot Sin(\theta_1 + \theta_2) \end{bmatrix} \tag{4.258}$$

Notice that when $\theta_2 = 0.0$, the mechanism is at a singular point, which is indicated by the $Sin(\theta_2)$ term in the denominator of the inverse Jacobian equation above.

4.8 Problems

1. Consider a gear reducer as shown in Fig. 4.1. Let the diameter of the gear on input shaft be equal to $d_1 = 2.0\ in$ and witdh $w_1 = 0.5\ in$. Assume that the gear is made of material steel and that it is a solid frame without any holes. The ouput gear has the same with and material, and the gear reduction from input to output is $N = 5$, ($d_2 = 10.0\ in$). The length and diameters of the shafts that extend to the sides of the gears are $d_{s1} = 1.0\ in,\ d_{s2} = 1.0\ in$ and $l_{s1} = 1.0\ in,\ l_{s2} = 1.0\ in$. Let us consider that there is a net load torque of $T_{L2} = 50 lb\ in$ at the output shaft.

1. Determine the net rotary inertia reflected on the input shaft due to two gears and two shafts.
2. Determine the necessary torque at the input shaft to balance the load torque.
3. If the input shaft is actuated by a motor that is controlled with 1/10 *degree* accuracy, what is the angular positioning accuracy that can be provided at the output shaft.

2. Repeat the same analysis and calculations for a belt and toothed pulley mechanism. The gear ratios and the shaft sizes are same. The load torque in the output shaft is same. Neglect the inertia of the belt. Comment on the functional similarities. Also discuss practical differences between the two mechanisms.

3. Consider a linear positioning system using a ball-screw mechanism. The ball-screw is driven by an electric servo motor. The ball screw is made of steel, has length of $l_{ls} = 40\ in$, and diameter of $d_{ls} = 2.5\ in$. The pitch of the lead is $p = 4\ rev/in$ (or the lead is $0.25\ in/rev$). Assume that the lead-screw mechanism is in vertical direction and moving a load of 100 *lbs* against the gravity up and down in z-direction.

1. Determine the net rotary inertia reflected on the input shaft of the motor.
2. Determine the necessary torque at the input shaft to balance the weight of the load due to gravity.
3. If the input shaft is actuated by a motor that is controlled with 1/10 *degree* accuracy, what is the translational positioning accuracy that can be provided at the output shaft.

4. For the problem 3, consider that the typical cyclic motion that the workpiece is to make is defined by a trapezoidal velocity profile. The load is to be moved a distance of 1.0 *in* in 300 *msec*, wait there for 200 *msec* and then move in the reverse direction. This motion is repeated continuously. Assume the the 300 *msec* motion time is equally divided between acceleration, run and deceleration times, $t_a = t_r = t_d = 100\ msec$ (Fig.4.14).

1. Calculate the necessary torque (maximum and continuous rated) and maximum speed required at the motor shaft. Select an appropriate servo motor for this application.
2. If a incremental position encoder is used on the motor shaft for control purposes, what is the required minimum resolution in order to provide a tool positioning accuracy of $40 \mu in$.

3. Select a proper flexible coupling for this application to be used between the motor shaft and the lead screw. Include the inertia of the flexible coupling in the inertia calculations and motor sizing calculations. Repeat step 1.

5. Repeat the same analysis and calculations of Problem 3 and 4, if a rack-pinion mechanism was used to convert the rotary motion of motor to a translational motion of the tool.

1. First, determine the rack and pinion gear that gives the same gear ratio (from rotary motion to translational motion) as the ball-screw mechanism.
2. Discuss the differences between the ball-screw, rack-pinion, and belt-pulley (translational version) mechanisms.

6. Given a four bar linkage (Fig.4.7), derive the geometric relationship between the following motion variables of the mechanism. Let the link lengths to be l_1, l_2, l_3, l_4.

1. Input is the angular position ($\theta_1(t)$) of link 1, output is the angular position of link 3, ($\theta_3(t)$). Find $\theta_3 = f(\theta_1)$. Plot θ_3 for one revolution of link 1 as function of θ_1.
2. Determine the x and y coordinates of the tip of the link 3 during the same motion cycle. Plot the results on the x-y plane (path of the tip of link 3 during one revolution of link 1).

7. Consider the cam and follower mechanism shown in Fig.4.9. The follower arm is connected to a spring. The follower is to make an up-down motion once per revolution of the cam. The travel range of the follower is to be 2.0 *in* total.

1. Select a modified trapezoidal cam profile for this task.
2. Assume the input shaft to the cam is driven at 1200 *rpm* constant speed. Calculate the maximum linear speed and linear accelerations experienced at the tool tip.
3. Let the stiffness of the spring be $k = 100 lb/in$ and the mass of the follower and the tool it is connected to $m_f = 10 lb$. Assume the input shaft motion is not affected by the dynamics of the follower and tool. The input shaft rotates at constant speed at 1200 $[rev/min]$. Determine the net force function at the follower and tool assembly during one cycle of the motion and plot the result. Notice that the

$$F(t) = m_f \ddot{x}(t) + k \cdot x(t) \tag{4.259}$$

and $x(t), \ddot{x}(t)$ are determined by the input shaft motion and cam function. What happens if the net force $F(t)$ becomes negative? One way to assume that $F(t)$ does not become negative is to use a preloaded spring. What is the preloading requirement to assure $F(t)$ is always positive during the planned motion cycle? Preload spring force can be taken into account in the above equation as follows,

$$F(t) = m_f \ddot{x}(t) + k \cdot x(t) + F_{pre} \tag{4.260}$$

where $F_{pre} = k \cdot x_o$ is a constant force due to the preloading of the spring. This force can be set to a constant value by selection of spring constant and initial compression.

8. Consider an electric servo motor and a load it drives through a gear reducer (Fig.4.15).

1. What is the generally recommended relationship between the motor inertia and reflected load inertia ?
2. What is the optimal relationship in terms of minimizing the heating of the motor ?
3. Derive the relationship for the optimal relationship.

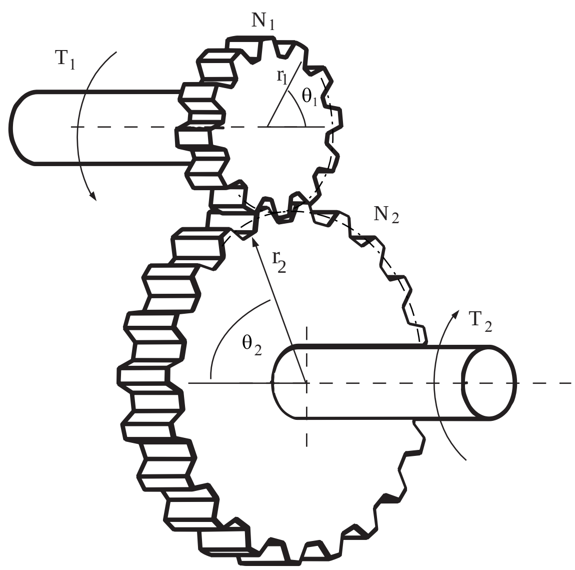


Figure 4.1: Rotary to rotary motion conversion mechanisms: gear mechanism.

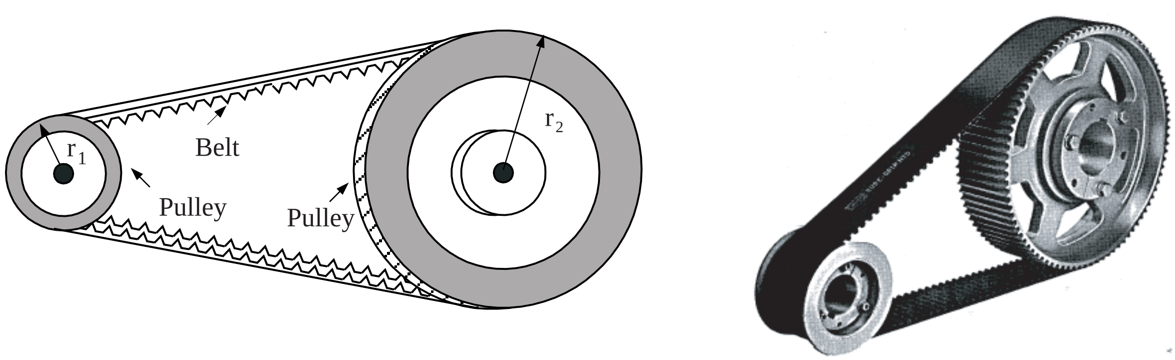


Figure 4.2: Rotary to rotary motion conversion mechanisms: timing belt and toothed pulley.

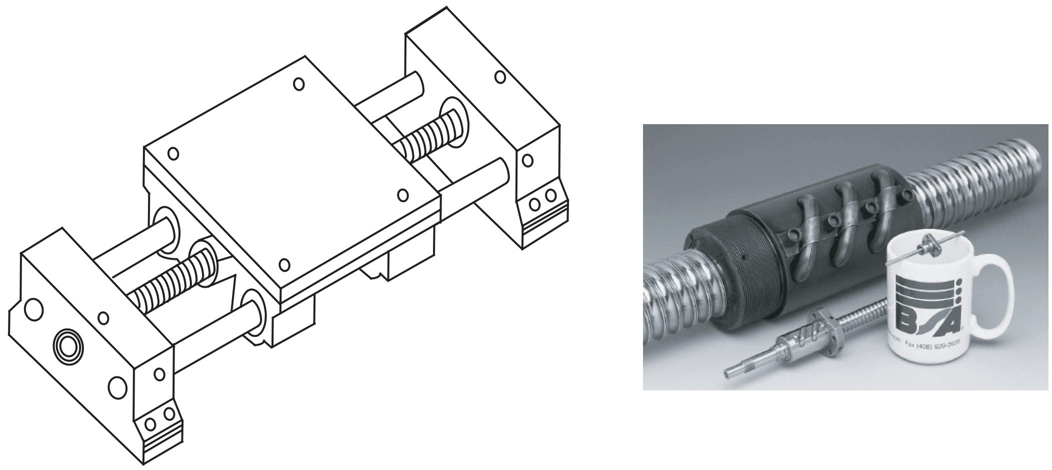


Figure 4.3: Rotary to translational motion conversion mechanism: lead-screw or ball-screw with linear guide bearings.

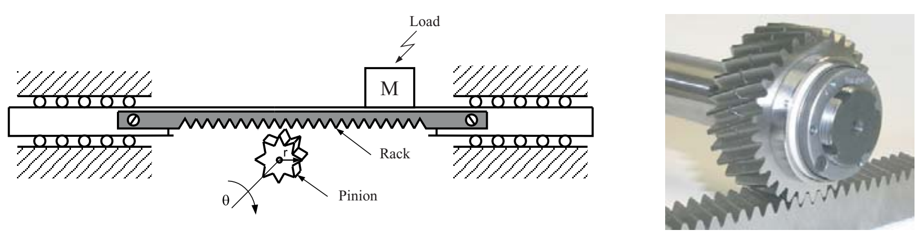


Figure 4.4: Rotary to translational motion conversion mechanism: rack and pinion mechanism. The advatange of the rack-pinion mechanism over the lead screw mechanism is that the translational motion range can be very long. The lead screw length is limited by the torsional stiffness. In rack-pinion mechanism since the translational part does not rotate, it does not have the reduced torsional stiffness problem due to the long length.

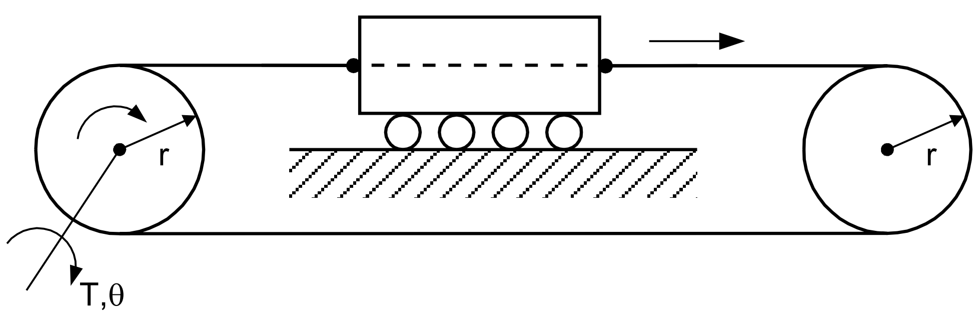


Figure 4.5: Rotary to translational motion conversion mechanism: Belt-pulley mechanism where both pulleys have the same diameter. The output motion is taken from the belt as the translational motion.

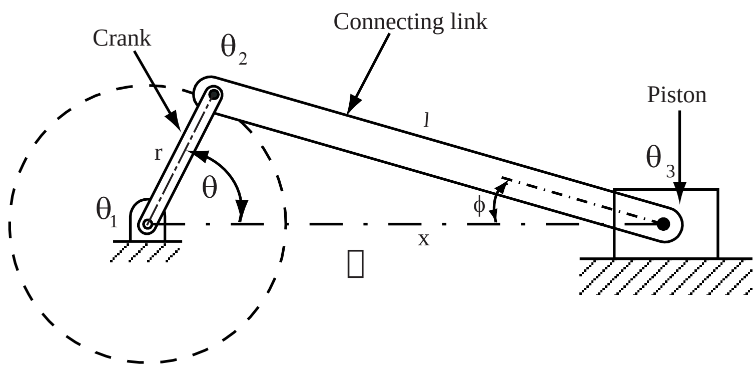


Figure 4.6: Rotary to translational motion conversion mechanism: slider-crank mechanism. In internal combustion engine, the mechanism is used as translational (piston motion is input) to rotary motion (crank shaft rotation) conversion mechanism.

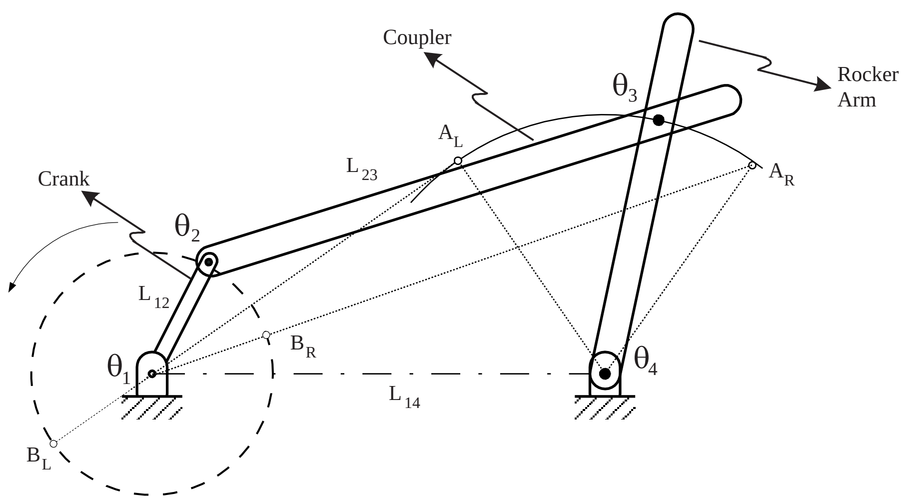


Figure 4.7: Four bar mechanism.

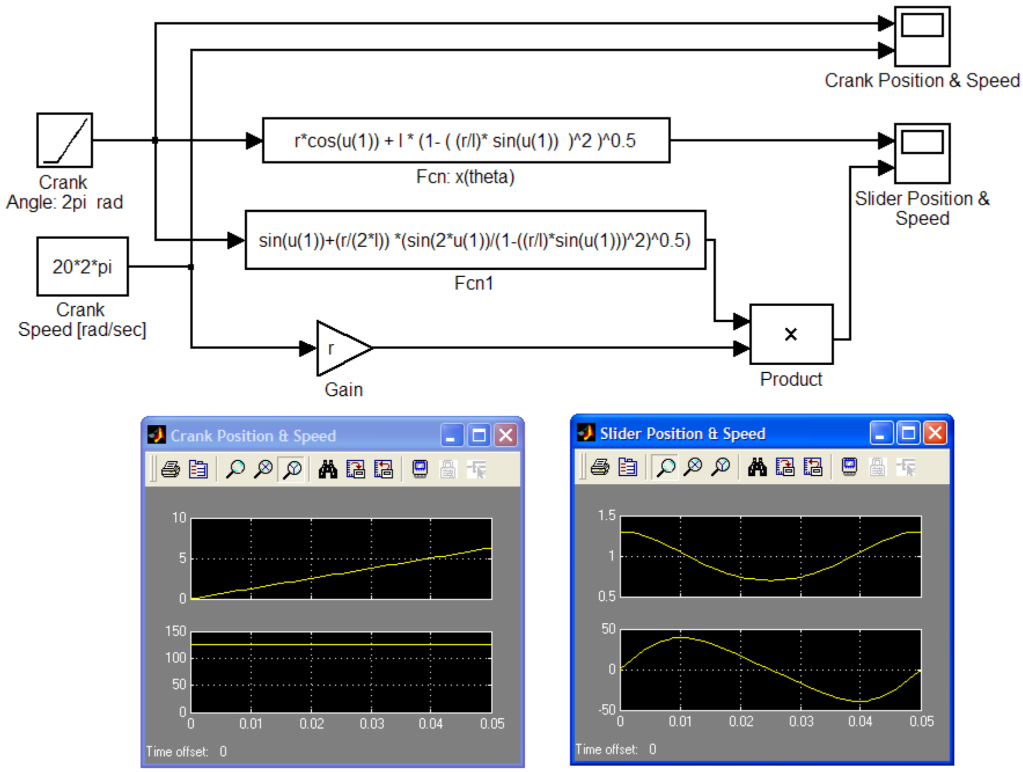


Figure 4.8: Simulation result of a slider crank mechanisms: $r = 0.3\ m$, $l = 1.0m$, speed of crank shaft is constant at $\dot{\theta} = 1200\ rpm$. The resulting slider position and speed functions are shown in the figure.

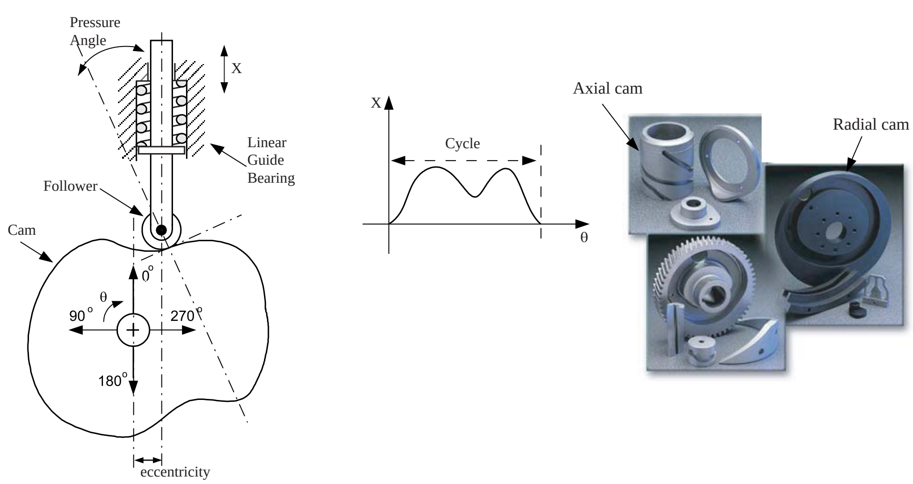


Figure 4.9: Rotary to translational motion conversion mechanism: cam mechanism.

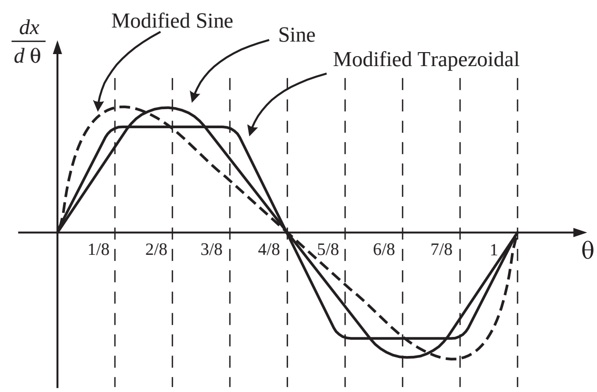


Figure 4.10: Commonly used cam profiles. The acceleration function is shown as function of the driving axis. Sinusoidal, modified sin, and modified trapezoidal functions are common cam profiles.

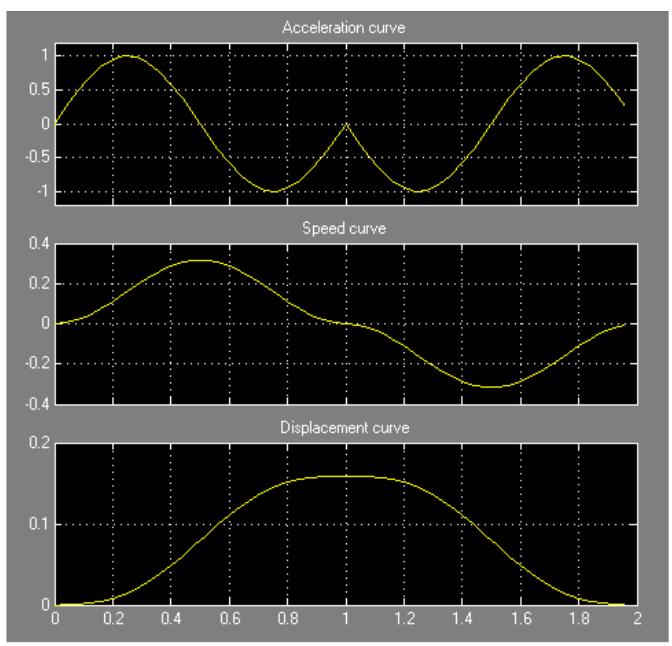


Figure 4.11: Sinusoidal acceleration cam profile: acceleration, speed and displacement function for a cam design where half of shaft revolution is used for rise motion and the other half is used for down motion symmetrically.

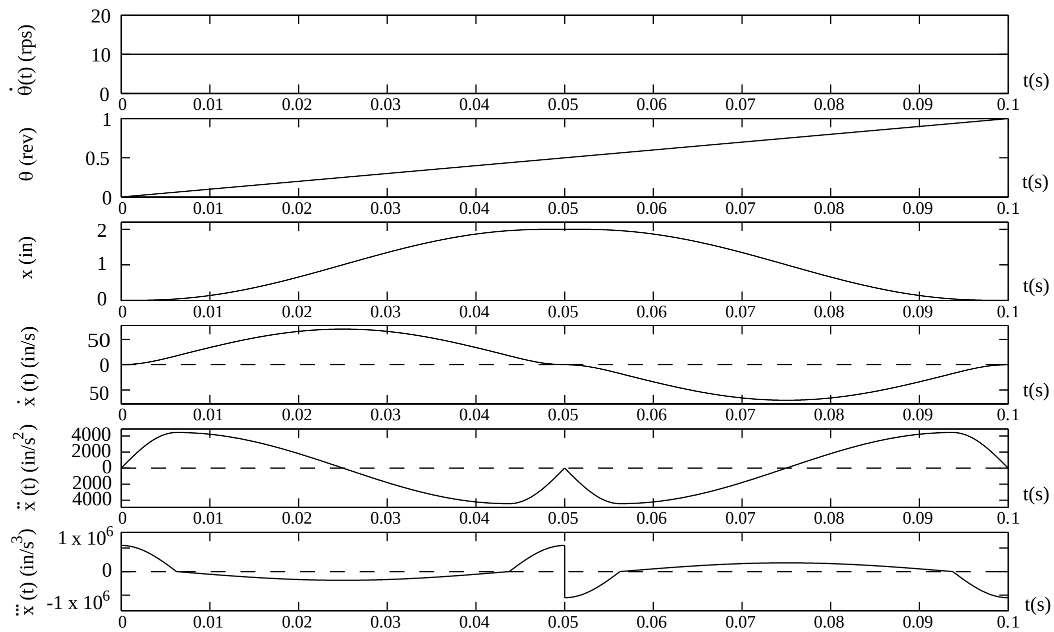


Figure 4.12: Modified sine cam function: displacement, speed, acceleration and jerk functions.

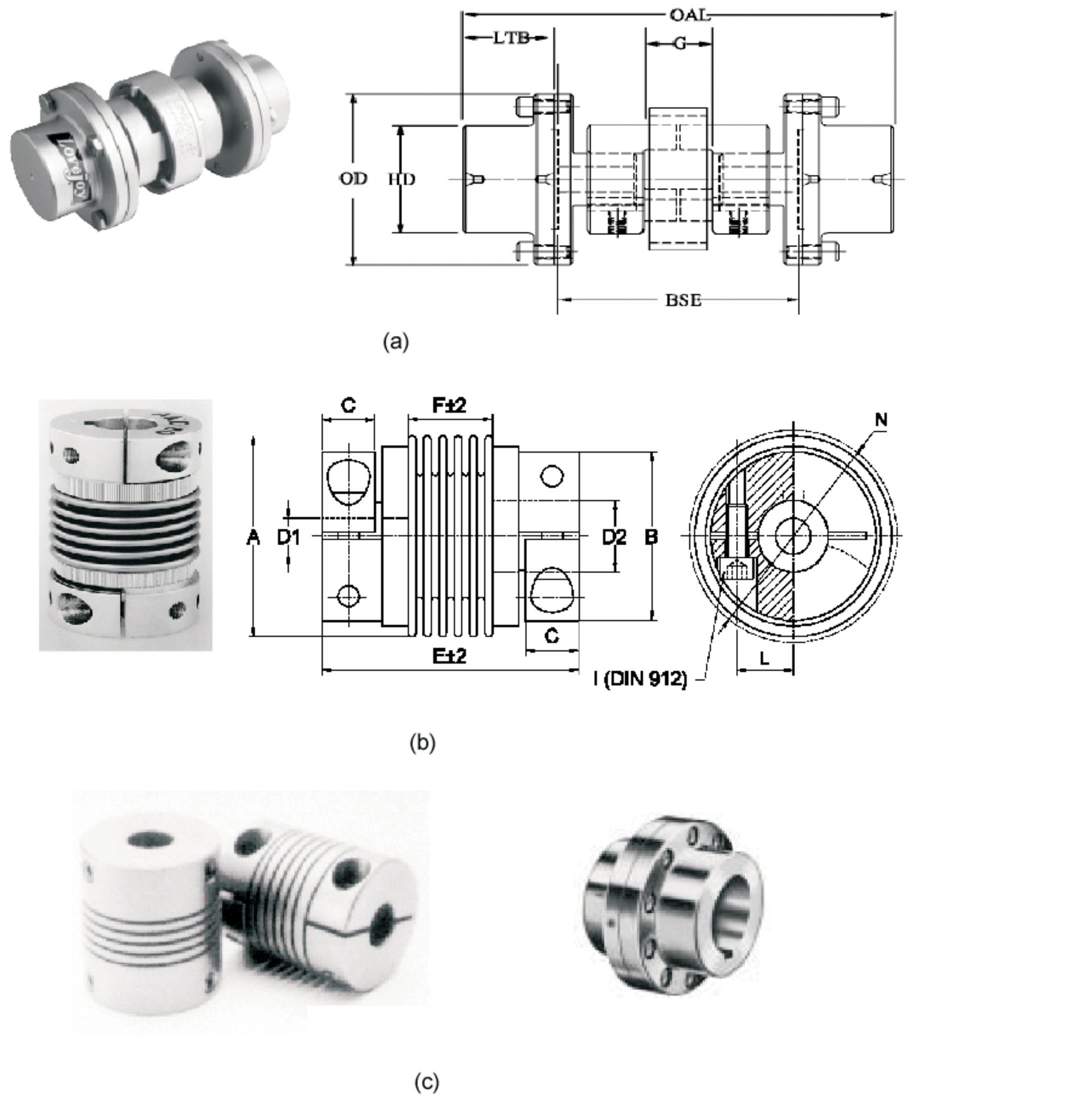


Figure 4.13: Flexible couplings used between connecting shafts in motion transmission mechanisms to compensate for the shaft misallignments and protect bearings.

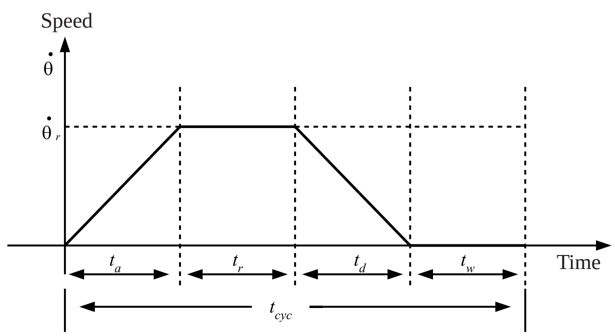


Figure 4.14: A typical desired velocity profile of a motion axis in programmable motion control applications such as automated assembly machines, robotic manipulators.

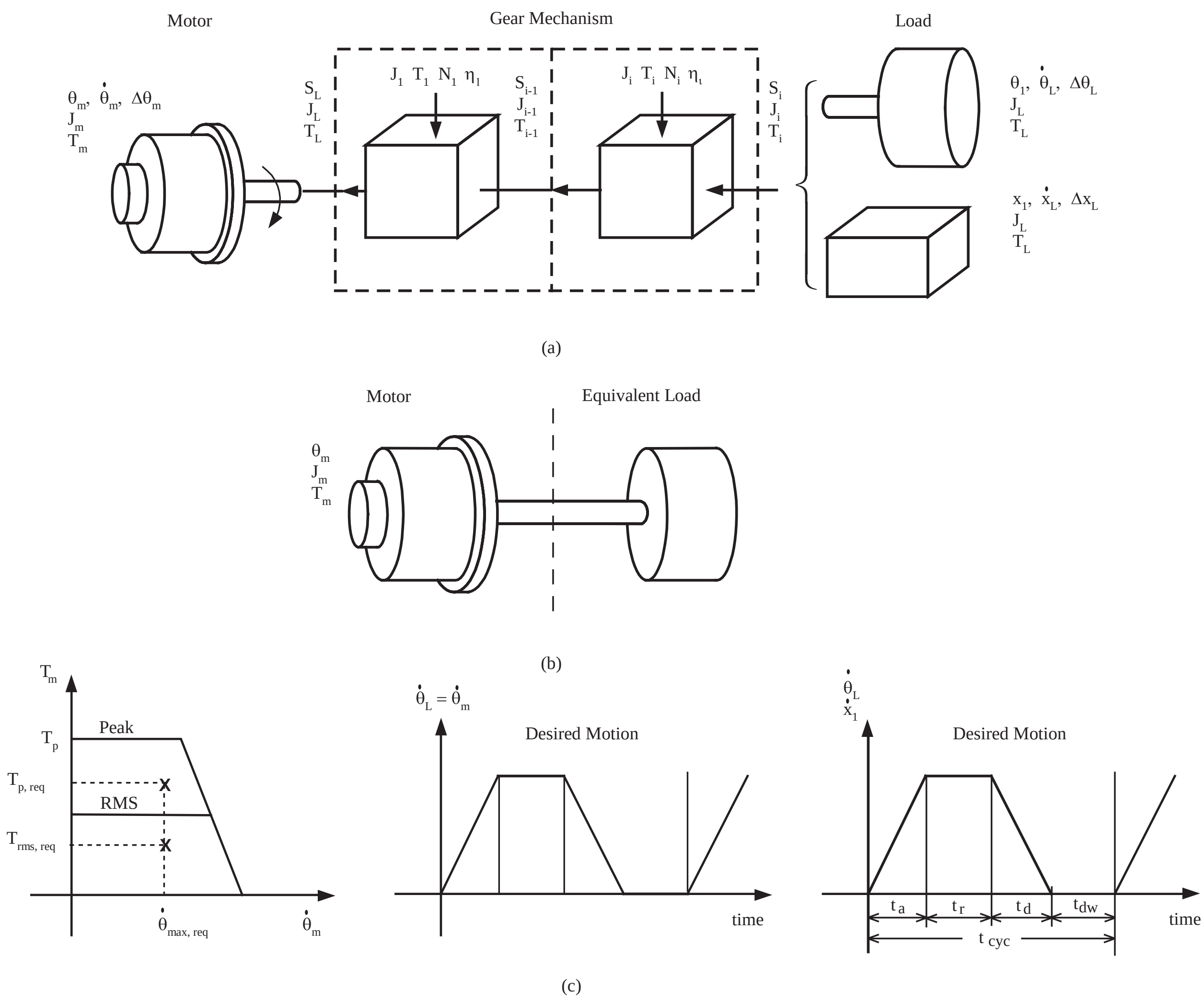


Figure 4.15: Actuator sizing: load (inertia, torque/force), gear mechanism, and desired motion must be specified.

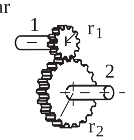
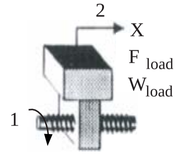
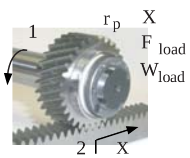
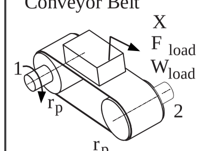
	Mechanism Characteristics				Input Characteristics			Output Characteristics		
Mechanism Type	n_1	η_1	J_1	T_1	S_2	J_2	T_2	S_1	J_1	T_1
Gear 	$\frac{r_2}{r_1}$	≤ 1.0	J_{r1} J_{r2}	$Tc.sgn(\dot{\theta}_1)$ $+c\,\dot{\theta}_1$	θ_2	J_2	T_2	$\theta_1 = n\,\theta_2$ $= (\frac{r_2}{r_1})\,\theta_2$	$J_{r1} + \left(\frac{J_{r2}}{n^2\eta}\right)$	$T_i + \frac{1}{n\eta} T_2$
Belt-Pulley 	$\frac{r_2}{r_1}$	≤ 1.0	J_{r1} J_{r2} W_{belt}	$Tc.sgn(\dot{\theta}_1)$ $+c\,\dot{\theta}_1$	θ_2	J_2	T_2	$\theta_1 = n\,\theta_2$ $= (\frac{r_2}{r_1})\,\theta_2$	$J_{r1} + \left(\frac{J_{r2} + J^2}{n^2\eta}\right)$ $+ \frac{1}{2}\left(\frac{W_{belt}}{g}\right)(r_1^2 + r_2^2)$	$T_i + \frac{1}{n\eta} T_2$
Ball Screw or Cam  $X=\theta/2\pi p$	$2\pi p$	≤ 1.0	J_{lead} W_{load}	$Tc.sgn(\dot{\theta}_1)$ $+c\,\dot{\theta}_1$	X	W_{load}	F_{load}	$\theta_1 = n\,X$ $= (2\pi p)\,X$	$J_{load} + \left(\frac{W_{load}}{g}\right)\left(\frac{1}{n^2\eta}\right)$	$T_i + \frac{1}{n\eta} F_{load}$
Rack Pinion 	$\frac{1}{r_p}$	≤ 1.0	J_{pinion} W_{rack}	$Tc.sgn(\dot{\theta}_1)$ $+c\,\dot{\theta}_1$	X	W_{load}	F_{load}	$\theta_1 = n\,X$ $= (\frac{1}{r_p})\,X$	J_{pinion} $+ \left(\frac{W_{rack}}{g}\right)\left(\frac{1}{n^2\eta}\right)$ $+ \left(\frac{W_{load}}{g}\right)\left(\frac{1}{n^2\eta}\right)$	$T_i + \frac{1}{n^2\eta} F_{load}$
Conveyor Belt 	$\frac{1}{r_p}$	≤ 1.0	J_{p1} J_{p2} W_{belt}	$Tc.sgn(\dot{\theta}_1)$ $+c\,\dot{\theta}_1$	X	W_{load}	F_{load}	$\theta_1 = n\,X$ $= (\frac{1}{r_p})\,X$	$J_{p1} + J_{p2}$ $+ \left(\frac{W_{belt} + W_{load}}{g}\right)\left(\frac{1}{n^2\eta}\right)$	$T_i + \frac{1}{n^2\eta} F_{load}$

Figure 4.16: Commonly used motion converstion (gear) mechanisms and their input-output relationships.

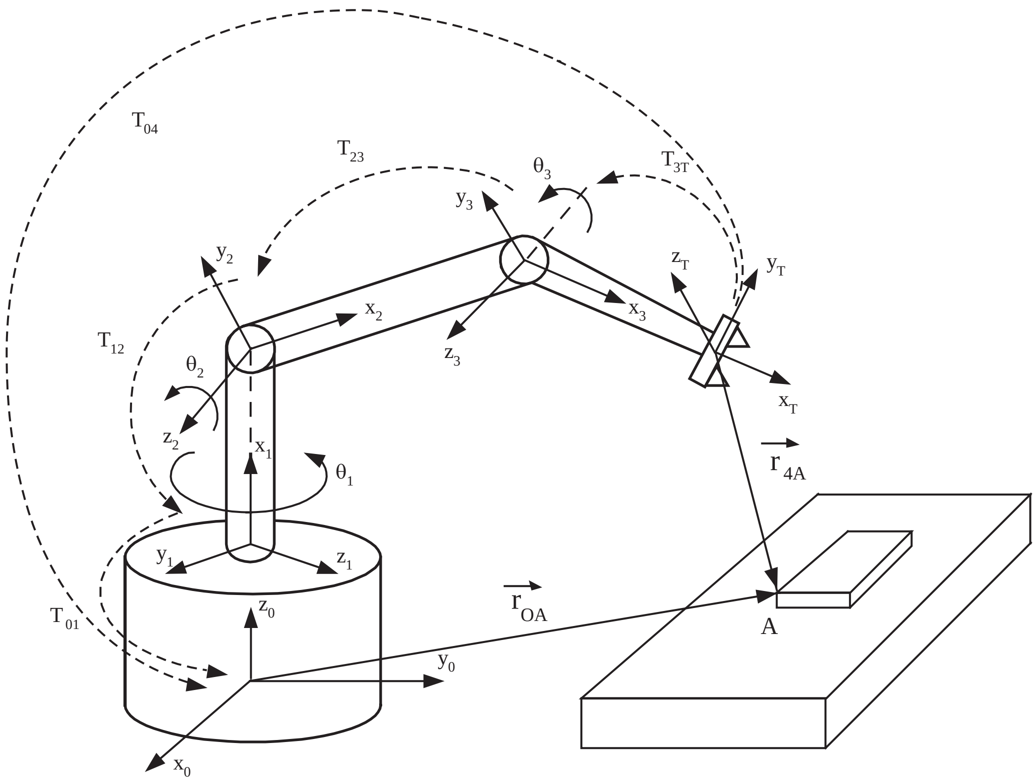


Figure 4.17: Multi-degree of freedom mechanisms: a robotic manipulator with three joints. We use this example to illustrate how to desribe the position and orientation of one coordinate frame with respect to another. If we attach a coordinate frame to a workpiece, we can describe the position and orientation of it with respect to other coordinate frames through the coordinate frame attached to it.

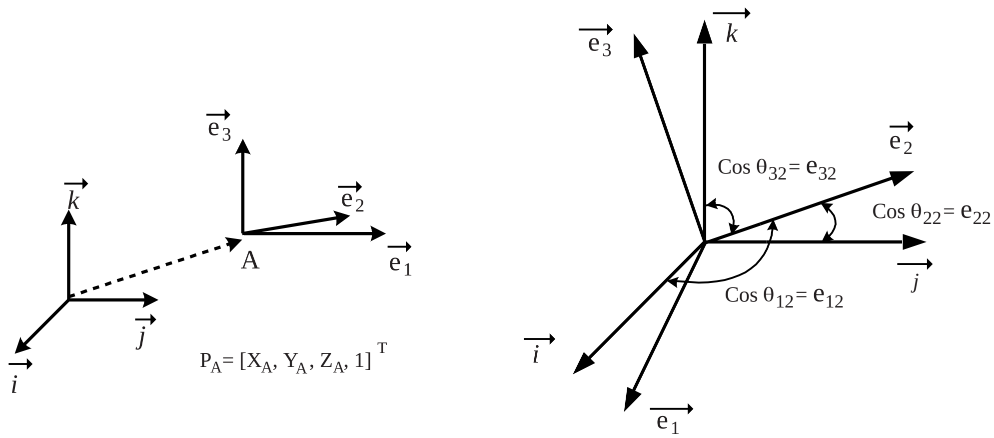


Figure 4.18: Use of (4x4) coordianate transformation matrices to describe the kinematic (geometric) relationships between different objects in three dimensional space. A point is described with respect to a reference coordinate frame by its three position coordinates. An object is described by a coordinate frame fixed to it: position of its origin and its orientation with respect to the reference coordinate frame.

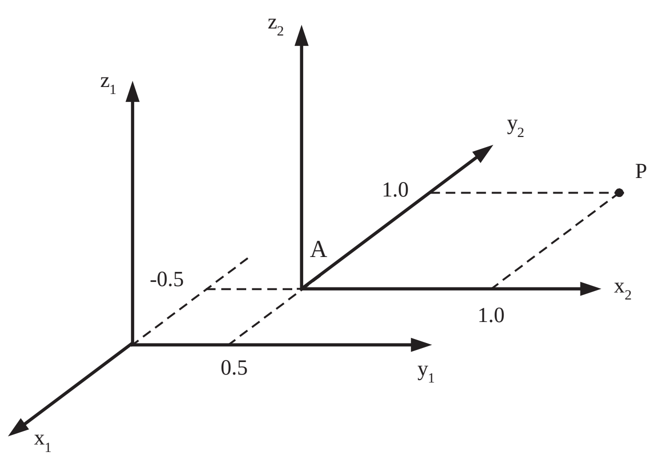


Figure 4.19: Describing the position and orientation of one coordinate frame with respect to other.

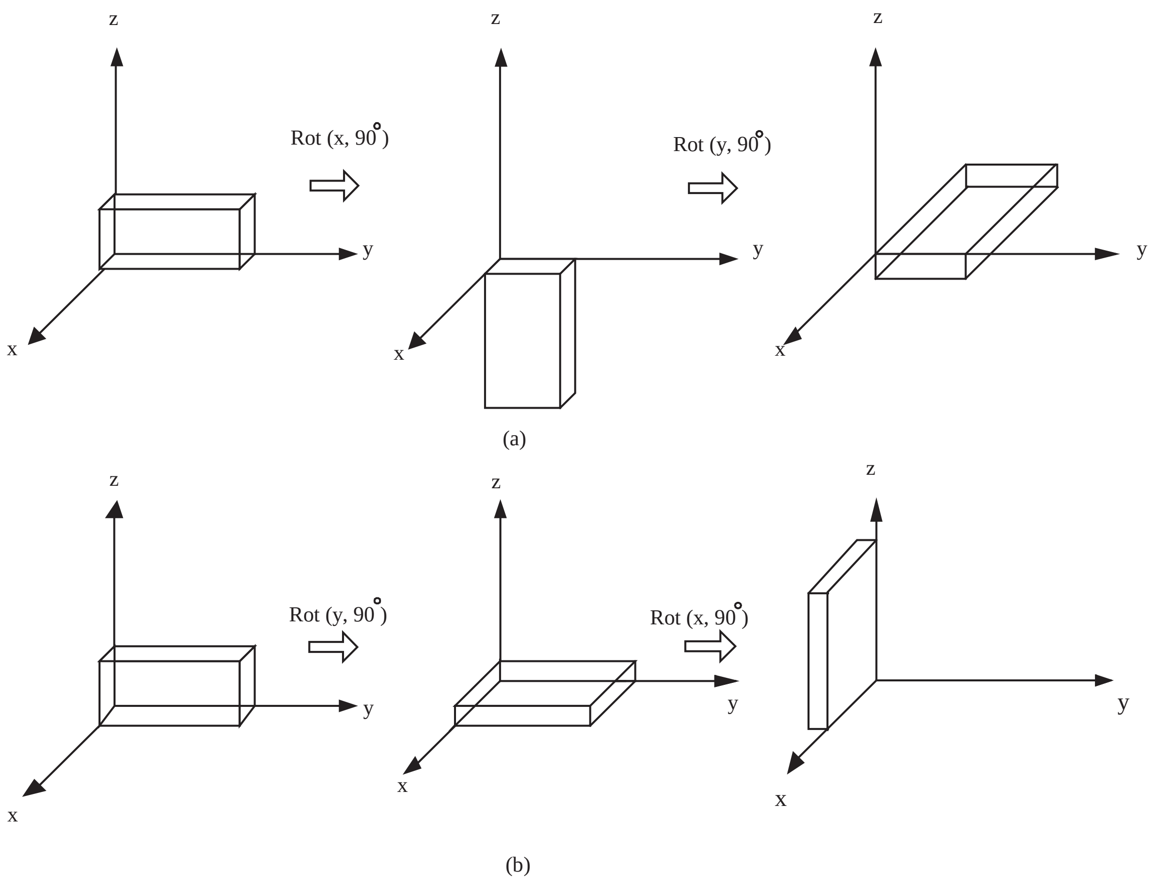


Figure 4.20: Two finite rotation sequence performed in reverse order. The final orientations are different. Finite rotations are not commutative.

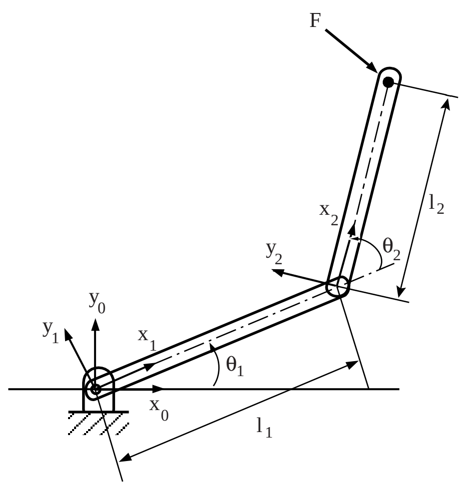


Figure 4.21: Kinematic description of a two joint planar robotic manipulator.